EE C245 – ME C218
Introduction to MEMS Design
Fall 2009

Prof. Clark T.-C. Nguyen

Dept. of Electrical Engineering & Computer Sciences
University of California at Berkeley
Berkeley, CA 94720

Lecture Module 9: Energy Methods

Lecture Outline

- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
    - Virtual Work
    - Energy Formulations
    - Tapered Beam Example
    - Estimating Resonance Frequency

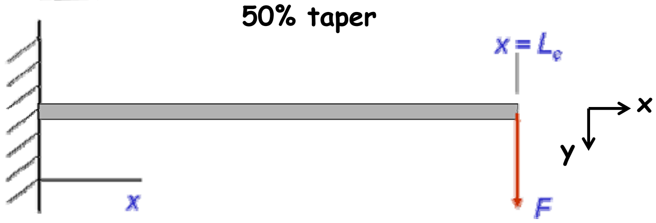
Energy Methods

More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)

Top view of cantilever's $W(x)$

$$W(x) = W(1 - \frac{x}{2L_c})$$



Solution: Use Principle of Virtual Work

UC Berkeley

- In an energy-conserving system (i.e., elastic materials), the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces …
- Implication: if we can formulate stored energy as a function of the deformation of a mechanical object, then we can determine how an object responds to a force by determining the shape the object must take in order to minimize the difference U between the stored energy and the work done by the forces:

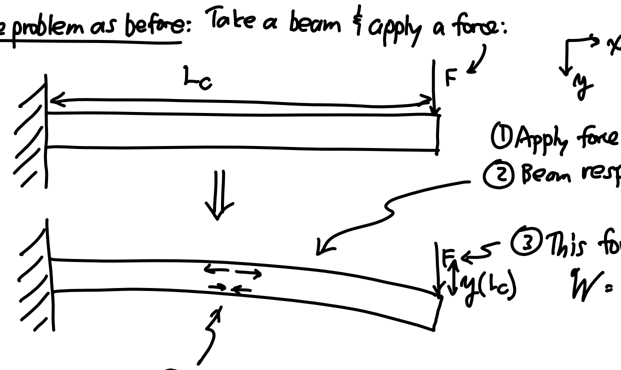
$$U = \text{Stored Energy} - \text{Work Done}$$

- Key idea: we don't have to reach U = 0 to produce a very useful, approximate *analytical* result for load-deflection

More Visual Description …

UC Berkeley

Same problem as before: Take a beam & apply a force:



① Apply force.

② Beam responds by bending.

③ This force has done work: $W = F \cdot y(L_c)$

④ Strain generated → This means the beam has received an influx of stored energy — magnitude determined by its deformed shape

⑤ Then:

$$U = \text{Stored Energy} - \text{Work Done} \rightarrow 0$$

When we choose the right shape! (This is how we get the beam's response to F!)

Fundamentals: Energy Density

UC Berkeley

- Strain energy density: [J/m^3]
  - To find work done in straining material

$W(Q) = \int_0^Q \frac{Q}{C} dQ \rightarrow$ charging a capacitor from $0 \rightarrow Q$ takes this much work $\rightarrow$ stored energy on a capacitor

This is a definition, so really can just say it's a definition.

$$w = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \quad \text{x-axis normal stress term}$$

$\varepsilon_x \leftarrow$ value of strain @ position (x,y,z)

$\sigma_x(\varepsilon_x) \rightarrow$ relates stress to strain @ position (x,y,z)

$$[\sigma_x = E\varepsilon_x] \Rightarrow w = \int_0^{\varepsilon_x} E\varepsilon_x d\varepsilon_x = \frac{1}{2}E{\varepsilon_x}^2$$

$W(q_1) = \int_0^{q_1} e(q)dq$ q = displacement, e = effect } Generic Definition of Work

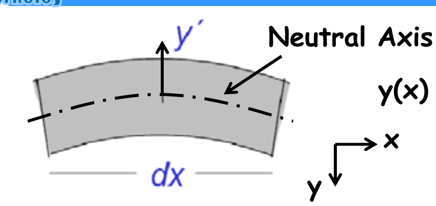
- Total strain energy [J]:
  - Integrate over all strains (normal and shear)

$$\mathcal{W} = \iiint\left(\frac{1}{2}E\left({\varepsilon_x}^2 + {\varepsilon_y}^2 + {\varepsilon_z}^2\right) + \frac{1}{2}G\left({\gamma_{xy}}^2 + {\gamma_{xz}}^2 + {\gamma_{yz}}^2\right)\right)dV$$

EE C245: Introduction to MEMS Design LecM 9 C. Nguyen 9/28/07 7

Bending Energy Density

UC Berkeley



y(x) = transverse displacement of neutral axis

- First, find the bending energy dW_{bend} in an infinitesimal length dx:

W = width

$$dW_{bend} = Wdx\int_{-h/2}^{h/2} \frac{1}{2}E\epsilon_x^2(y')dy'$$

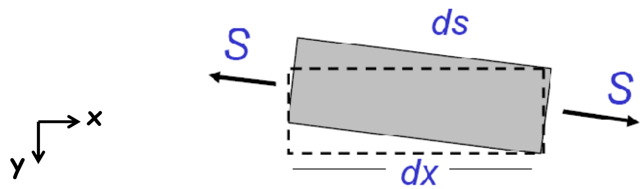
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y) = y'\frac{d^2y}{dx^2}$$

$$dW_{bend} = W dx\int_{-h/2}^{h/2} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy' = \frac{1}{2}E\overbrace{\left(\frac{Wh^3}{12}\right)}^{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2}EI_z\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

EE C245: Introduction to MEMS Design LecM 9 C. Nguyen 9/28/07 8

Energy Due to Axial Load



- Strain due to axial load S contributes an energy $d\mathcal{W}_{stretch}$ in length dx, since lengthening of the different element dx (to ds) results in a strain ε_x

Binomial Theorem

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \approx dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \varepsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

Axial Strain Energy

$$\left[d\mathcal{W}_{axial} = S\varepsilon_x dx = \frac{1}{2}S\left(\frac{dy}{dx}\right)^2 dx\right] \Rightarrow \boxed{\mathcal{W}_{axial} = \frac{1}{2}S\int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Shear Strain Energy

$$\boxed{\mathcal{W}_{shear} = \frac{3(EI_z)^2}{4GWh}\int_0^L \left(\frac{d^3y}{dx^3}\right)^2 dx}$$

Shear Modulus

- See W.C. Albert, "Vibrating Quartz Crystal Beam Accelerometer," Proc. ISA Int. Instrumentation Symp., May 1982, pp. 33-44

Applying the Principle of Virtual Work

UC Berkeley

- Basic Procedure:
  - Guess the form of the beam deflection under the applied loads
  - Vary the parameters in the beam deflection function in order to minimize:

Assumes point load

$$U = \underbrace{\sum_j W_j}_{\text{Sum strain energies}} - \sum_i F_i u_i$$

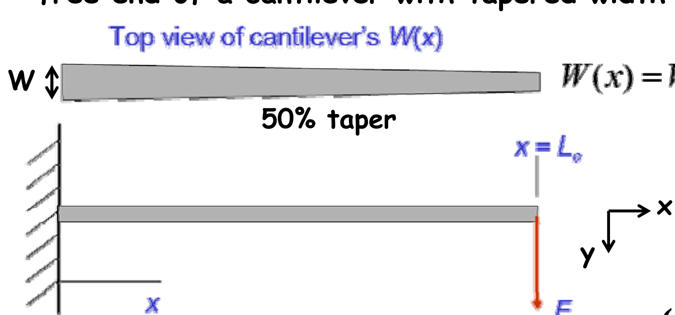
Displacement at point load

  - Find minima by simply setting derivatives to zero
- See Senturia, pg. 244, for a general expression with distrubuted surface loads and body forces

Example: Tapered Cantilever Beam

UC Berkeley

- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



$$W(x) = W(1 - \frac{x}{2L_c})$$

Adjustable parameters: minimize U

- Start by guessing the solution $\rightarrow$ $y(x) = c_2 x^2 + c_3 x^3$
  - It should satisfy the boundary conditions
  - The strain energy integrals shouldn't be too tedious
    - This might not matter much these days, though, since one could just use matlab or mathematica

Strain Energy And Work By F

$$U = \mathcal{W}_{bend} - F \cdot y(L_c)$$

$$\mathcal{W}_{bend} = \frac{1}{2} E \int_0^{L_c} I_z(x) \left(\frac{d^2 y}{dx^2} \right)^2 dx$$ (Bending Energy)

$$I_z(x) = \frac{W(x) h^3}{12}$$

$$\frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x$$

(Using our guess)

$$W(x) = W\left(1 - \frac{x}{2L_c}\right)$$

$$= \frac{1}{24} E W h^3 \int_0^{L_c} \left(1 - \frac{x}{2L_c}\right) (2c_2 + 6c_3 x)^2 dx - F \underbrace{\left(c_2 L_c^2 + c_3 L_c^3\right)}_{\text{Tip Deflection}}$$

Find c_2 and c_3 That Minimize U

- Minimize U → basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)
- The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respective to them are zero:

$$\frac{\partial U}{\partial c_2} = 0 \qquad \frac{\partial U}{\partial c_3} = 0$$

- Proceed:
  - First, evaluate the integral to get an expression for U:

$$U = E W h^3 \left\{ \frac{5 c_3^2}{16} L_c^3 + \frac{c_2 c_3}{3} L_c^2 + \frac{c_2^2}{8} L_c \right\} - F\left(c_2 L_c^2 + c_3 L_c^3\right)$$

Minimize U (cont)

- Evaluate the derivatives and set to zero:

$$\frac{\partial U}{\partial c_2} = 0 = \left(\frac{EWh^3}{3}c_3 - F\right)L_c^{\,2} + \left(\frac{EWh^3}{4}c_2\right)L_c$$

$$\frac{\partial U}{\partial c_3} = 0 = \left(\frac{5}{8}EWh^3c_3 - F\right)L_c^{\,3} + \left(\frac{EWh^3}{3}c_2\right)L_c^{\,2}$$

- Solve the simultaneous equations to get c_2 and c_3:

$$\boxed{c_2 = \left(\frac{84}{13}\right)\frac{FL_c}{EWh^3} \qquad c_3 = -\left(\frac{24}{13}\right)\frac{F}{EWh^3}}$$

The Virtual Work-Derived Solution

- And the solution:

$$y(x) = \left(\frac{24F}{13EWh^3}\right)\left(\left(\frac{7}{2}\right)L_c - x\right)x^2$$

- Solve for tip deflection and obtain the spring constant:

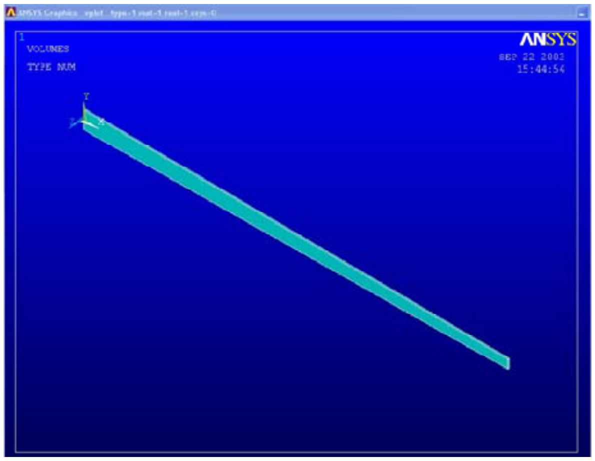
$$y(L_c) = \left(\frac{24F}{13EWh^3}\right)\left(\frac{5}{2}\right)L_c^{\,3} \qquad k_c = F / y(L_c) = \left(\frac{13EWh^3}{60L_c^{\,3}}\right)$$

- Compare with previous solution for constant-width cantilever beam (using Euler theory):

$$y(L_c) = \left(\frac{4F}{EWh^3}\right)L_c^{\,3} \longrightarrow$$ 13% smaller than tapered-width case

Comparison With Finite Element Simulation

- Below: ANSYS finite element model with
  - $L = 500\ \mu m$ $\quad W_{base} = 20\ \mu m$ $\quad E = 170\ GPa$
  - $h = 2\ \mu m$ $\quad W_{tip} = 10\ \mu m$



- Result: (from static analysis)
  - ↳ $k = 0.471\ \mu N/m$
- This matches the result from energy minimization to 3 significant figures

Need a Better Approximation?

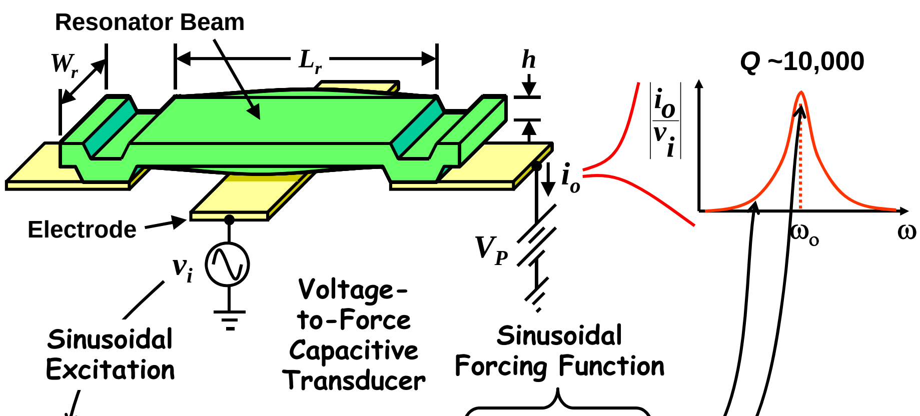
- Add more terms to the polynomial
- Add other strain energy terms:
  - ↳ Shear: more significant as the beam gets shorter
  - ↳ Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
  - ↳ Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
  - ↳ Can compare the importance of different terms
  - ↳ Should use in tandem with FEA for design

UC Berkeley

Estimating Resonance Frequency

UC Berkeley

Clamped-Clamped Beam μResonator

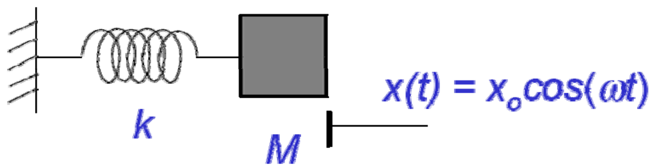


$$v_i = V_i \cos[\omega_o t] \longrightarrow f_i = \overbrace{F_i \cos[\omega_o t]}$$

- $\omega \neq \omega_o$: small amplitude
- $\omega = \omega_o$: maximum amplitude → beam reaches its maximum potential and kinetic energies

Estimating Resonance Frequency

- Assume simple harmonic motion:



- Potential Energy:

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_o^{\,2}\cos^2(\omega t)$$

- Kinetic Energy:

$$K(t) = \frac{1}{2}M\dot{x}^2(t) = \frac{1}{2}Mx_o^{\,2}\omega^2\sin^2(\omega t)$$

EE C245: Introduction to MEMS Design LecM 9 C. Nguyen 9/28/07 21

Estimating Resonance Frequency (cont)

- Energy must be conserved:
  - Potential Energy + Kinetic Energy = Total Energy
  - Must be true at every point on the mechanical structure

Occurs at peak displacement

Occurs when the beam moves through zero displacement

$$W_{max} = \frac{1}{2}kx_o^{\,2} = K_{max} = \frac{1}{2}M\omega^2 x_o^{\,2}$$

Maximum Potential Energy

Stiffness

Displacement Amplitude

Maximum Kinetic Energy

Mass

Radian Frequency

- Solving, we obtain for resonance frequency:

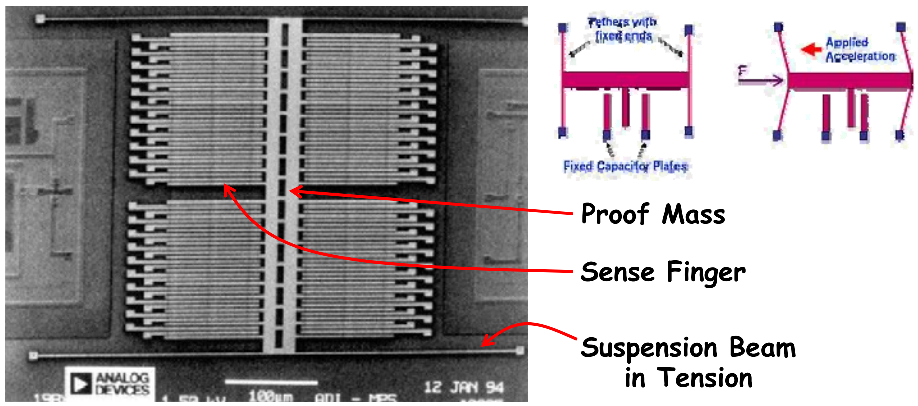
$$\boxed{\omega = \sqrt{\frac{k}{M}}}$$

EE C245: Introduction to MEMS Design LecM 9 C. Nguyen 9/28/07 22

UC Berkeley

Example: ADXL-50

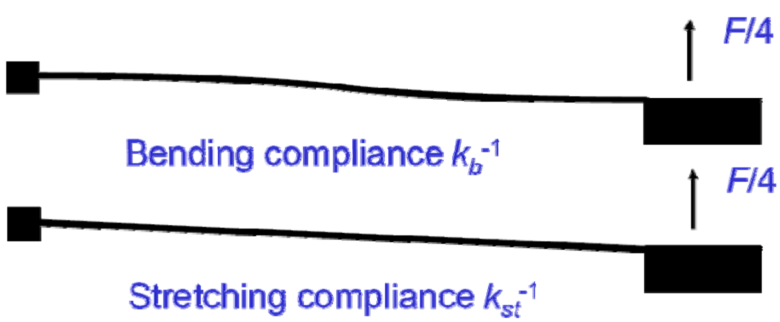
- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - ➯ Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



UC Berkeley

Lumped Spring-Mass Approximation

- Mass is dominated by the proof mass
  - ➯ 60% of mass from sense fingers
  - ➯ Mass = M = 162 ng (nano-grams)
- Suspension: four tensioned beams
  - ➯ Include both bending and stretching terms [A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998]



ADXL-50 Suspension Model

- Bending contribution:

$$k_b^{-1} = (1/k_c + 1/k_c) = 2\left[\frac{(L/2)^3}{3E(Wh^3/12)}\right] = \frac{L^3}{EWh^3} = 4.2\mu m/\mu N$$

- Stretching contribution:

$$k_{st}^{-1} = L/S = \frac{L}{\sigma_r Wh} = 1.14\mu m/\mu N$$

$$F_y = S\sin\theta \approx S(x/L) = \underbrace{\left(\frac{S}{L}\right)}_{= k_{st}} x$$

- Total spring constant: *add* bending to stretching (since they are in parallel)

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\mu N/\mu m$$

ADXL-50 Resonance Frequency

- Using a lumped mass-spring approximation:

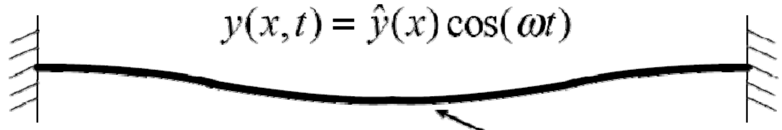
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{4.48N/m}{162x10^{-12}kg}} = 26.5kHz$$

- On the ADXL-50 Data Sheet: f_o = 24 kHz
  - Why the 10% difference?
  - Well, it's approximate … plus …
  - Above analysis does not include the frequency-pulling effect of the DC bias voltage across the plate sense fingers and stationary sense fingers … something we'll cover later on …

Distributed Mechanical Structures

- Vibrating structure displacement function:

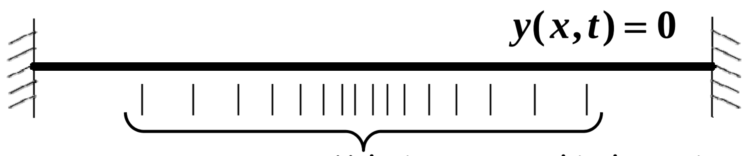
$$y(x,t) = \hat{y}(x)\cos(\omega t)$$



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Maximum Kinetic Energy

- Displacement: $y(x,t) = \hat{y}(x)\cos[\omega t]$
- Velocity: $v(x,t) = \dfrac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin[\omega t]$
- At times t = $\pi/(2\omega)$, $3\pi/(2\omega)$, …



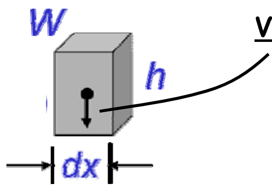
  - The displacement of the structure is y(x,t) = 0
  - The velocity is maximum and all of the energy in the structure is kinetic (since W=0):

$$v(x, \pi/(n\omega)) = -\omega\hat{y}(x)$$

Maximum Kinetic Energy (cont)

- At times t = π/(2ω), 3π/(2ω), …

$$y(x,t) = 0$$



Velocity: $v(x, \pi/(n\omega)) = -\omega \hat{y}(x)$

$$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$$

$$dm = \rho(Wh \cdot dx)$$

- Maximum kinetic energy:

$$K_{max} = \int_0^L \frac{1}{2} \rho W h dx v^2(x,t') = \int_0^L \frac{1}{2} \rho W h \omega^2 \hat{y}^2(x) dx$$

The Raleigh-Ritz Method

- Equate the maximum potential and maximum kinetic energies:

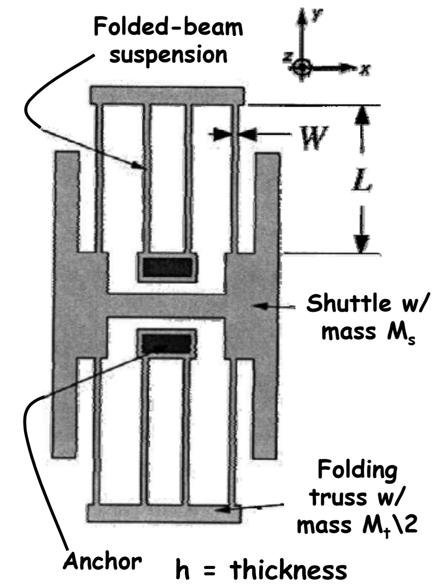
$$K_{max} = \int_0^L \frac{1}{2} \rho W h \omega^2 \hat{y}^2(x) dx = \mathcal{W}_{max}$$

- Rearranging yields for resonance frequency:

$$\boxed{\omega = \sqrt{\frac{\mathcal{W}_{max}}{\int_0^L \frac{1}{2} \rho W h \hat{y}^2(x) dx}}}$$

ω = resonance frequency
$\mathcal{W}_{max}$ = maximum potential energy
ρ = density of the structural material
W = beam width
h = beam thickness
ŷ(x) = resonance mode shape

Example: Folded-Beam Resonator



- Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh-Ritz method.

$$KE_{max} = PE_{max}$$

Kinetic Energy:

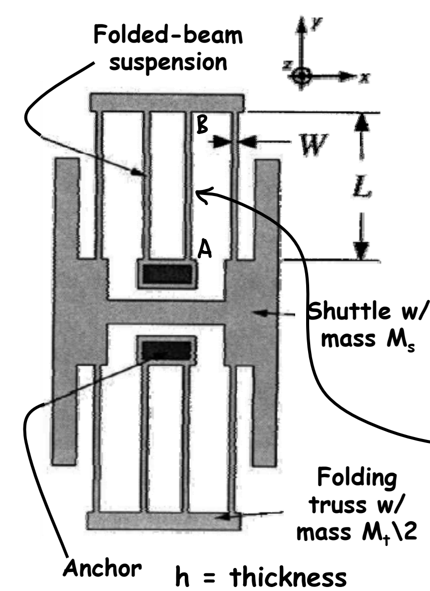
$$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{truss}} + \underbrace{KE_b}_{\text{beams}}$$

$$= \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \underbrace{\frac{1}{2}\int v_b^2 \, dM_b}$$

mass of both trusses

Must integrate since the beam velocity is a function of location y!

Get Kinetic Energies



Velocity of the shuttle: $v_s = \omega_0 X_0$

Resonance Freq.

Maximum Displacement Amplitude

$$\therefore \; KE_s = \frac{1}{2} v_s^2 M_s = \frac{1}{2}\omega_0^2 X_0^2 M_s$$

Velocity of the truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_0 X_0$

$$\therefore \; KE_t = \frac{1}{2}\left(\frac{1}{2}\omega_0 X_0\right)^2 M_t = \frac{1}{8}\omega_0^2 X_0^2 M_t$$

Velocity of the beam segments:

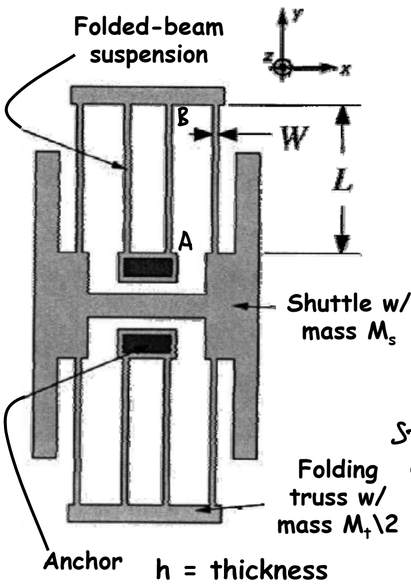
⇒ assume the mode shape is the same as the static displacement shape

⇒ For segment AB:

$$\hat{x}(y) = \frac{F_x}{48 E I_z}\left(3Ly^2 - 2y^3\right) \quad 0 \le y \le L \quad (1)$$

Get Kinetic Energies (cont)

UC Berkeley



At $y=L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$

Substituting into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for KE_b:

$$KE_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

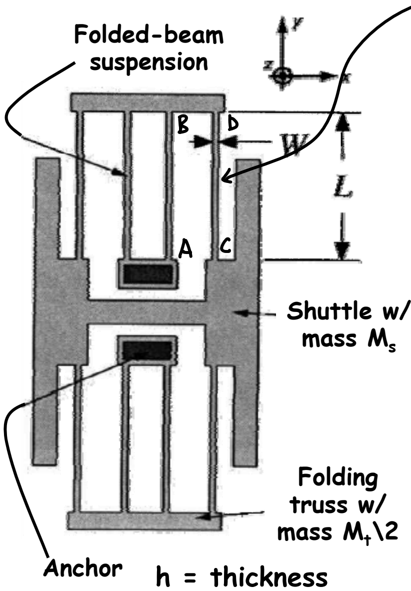
Static mass of beam [AB]

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$$KE_{[AB]} = \frac{13}{280} X_0^2\omega_0^2 M_{[AB]}$$

Get Kinetic Energies (cont)

UC Berkeley



For segment CD:

$$v_b(y)\Big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

Thus:

$$KE_{[CD]} = \frac{X_0^2\omega_0^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right] dy$$

$$KE_{[CD]} = \frac{83}{280} X_0^2\omega_0^2 M_{[CD]}$$

Static mass of beam [CD]

Let $M_b \triangleq$ total mass of the 8 beams.

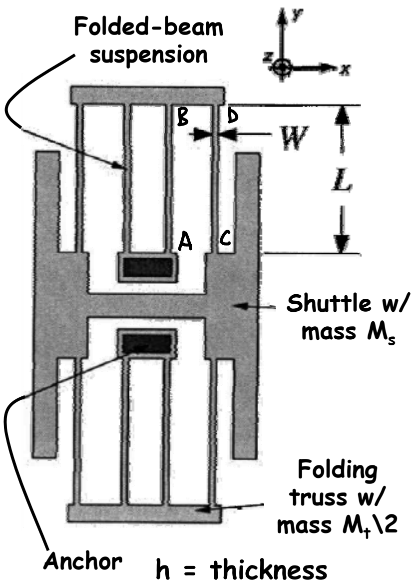
Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8}M_b$

Thus: $KE_b = 4KE_{[AB]} + 4K_{[CD]} = \frac{6}{35}X_0^2\omega_0^2 M_b$

and

$$KE_{max} = X_0^2\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]$$

Get Potential Energy & Frequency



PE_{max} is simply the work done to achieve maximum deflection:

$$PE_{max} = \frac{1}{2}\overset{=k_c}{k_x} X_0^2$$

Thus, using Raleigh-Ritz:

$$KE_{max} = PE_{max}$$

$$\cancel{X_0^2}\,\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}\overset{=k_c}{k_x}\cancel{X_0^2}$$

$$\boxed{\omega_0 = \left[\frac{\overset{=k_c}{k_x}}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

(Resonance Frequency of a Folded-Beam Suspended Shuttle)