Lecture 18: Energy Methods

- Announcements:
- Midterm is nearing: Thursday, Oct. 27
  - I passed out materials associated with the midterm, including and information sheet and old exams, last Tuesday
- HW#5 due today
  - Solutions will be posted tonight (or emailed)
- My office hours right after class
  - No office hours for me on Wednesday (since I'll be traveling)
  - But there are extra TA office hours
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- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
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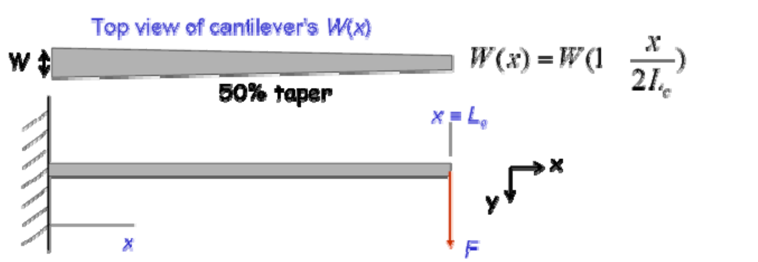
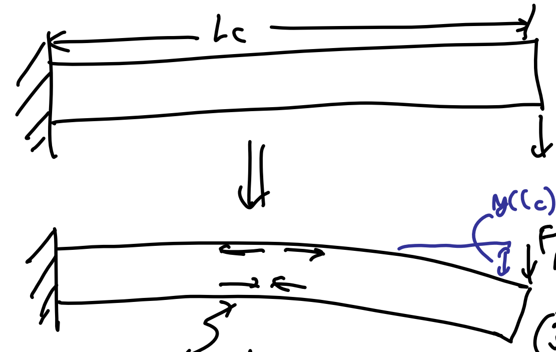
- Last Time:

- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



Same problem as before: Take a beam, apply a force.



① Apply force.

② Beam responds by bending.

③ This force has done work: $W = F \cdot y(L_c)$

④ Strain generated
- So the beam has received an influx of stored energy
- magnitude of " " determined by shape

⑤ Then

U = Stored Energy − Work Done → 0

when we choose the right shape.

This is how we got the beam's response to F.

transfer fcn

$y(x) = f(x)$

Fundamentals: Energy Density

General Definition of Work:

$$W(q_1) = \int_0^{q_1} e(q)\,dq$$

q = displacement
e = effort

for EE: $W(Q) = \int_0^{Q} \frac{Q}{C}\,dQ$

Strain Energy Density

$$w = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x$$

ϵ_x ← value of strain @ position (x,y,z)

$\sigma_x(\epsilon_x)$ → relates stress to strain @ position (x,y,z)

$[\sigma_x = E\epsilon_x]$

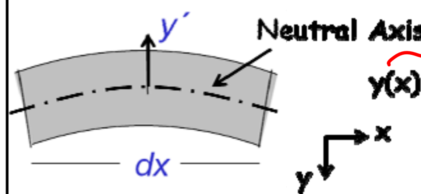
$$w = \int_0^{\epsilon_x} E\epsilon_x \, d\epsilon_x = \frac{1}{2} E \epsilon_x^2$$

Total Strain Energy: [J]

$$W = \iiint \left(\frac{1}{2} E\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + \frac{1}{2} G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right) dV$$

volume

Bending Energy Density



same as before

y(x) = transverse displacement of neutral axis

First, find the bending energy dW_{bend} in an infinitesimal length dx:

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \epsilon_x^2(y')\,dy'$$

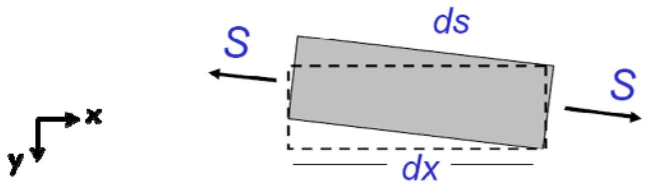
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y' \frac{d^2y}{dx^2}$$

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \left[y' \frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2} E \underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z} \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2} E I_z \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

Energy Due to Axial Load



⇒ energy related to lengthening:

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

binomial theorem

$$\simeq dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \ \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[\ dW_{axial} = S\epsilon_x dx = \frac{1}{2} S\left(\frac{dy}{dx}\right)^2 dx\ \right]$$

$$\boxed{W_{axial} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Axial Strain Energy

⇒ Look @ shear strain energy in your module.

- Go through Module 9 pages 10-18.

Estimating Resonance Frequency

$\mapsto x(t) = X_0 \cos\omega t$

Potential Energy

$$W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k X_0^2 \cos^2\omega t$$

Kinetic Energy

$$K(t) = \frac{1}{2} M \dot{x}^2(t) = \frac{1}{2} M X_0^2 \omega^2 \sin^2\omega t$$

$$\dot{x} = \frac{dx}{dt} = \text{velocity}$$

Remarks.

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$W_{max} = \frac{1}{2} k X_0^2 = K_{max} = \frac{1}{2} M \omega^2 X_0^2$$

maximum potential energy

peak displacement

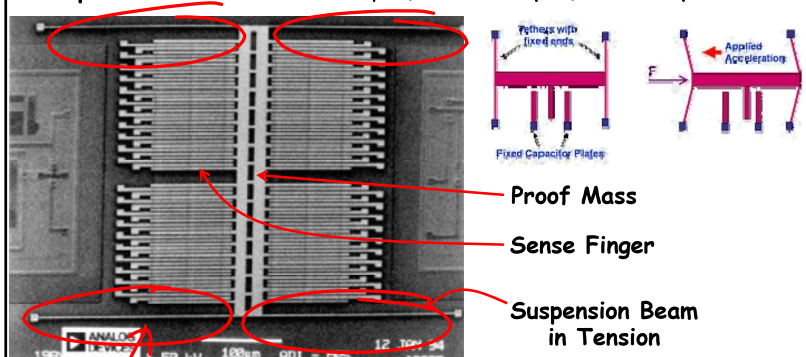
maximum kinetic energy

radian frequency

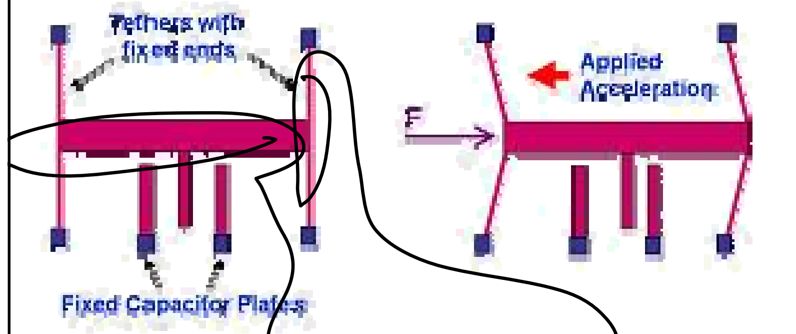
*

* — $\boxed{\omega_o = \sqrt{\frac{k}{M}}}$ ⇒ good for problems where mass & stiffness can be separated, i.e., are distinct

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - ↳ Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication: purposely introduce a $\underbrace{\text{tensile stress}}_{\text{a large one!}}$ in the beams!

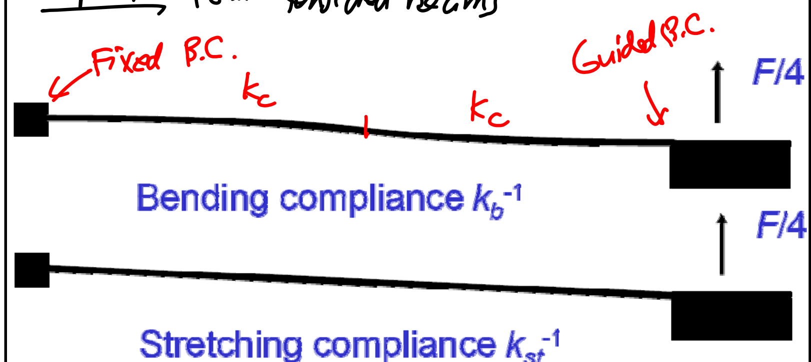


mass of structure >> mass of springs
∴ ignore the mass of the springs

stiffness of springs << stiffness of structure
∴ ignore the stiffness of the structure

For the ADXL-50: 60% of the mass from sense fingers → M = 162 ng

Suspension: Four tensioned beams

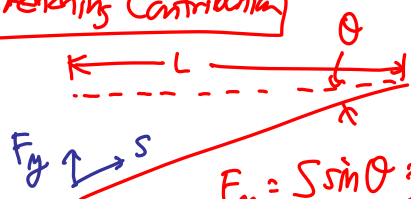


Bending Contribution

$$k_b^{-1} = \left(\frac{1}{k_c}+\frac{1}{k_c}\right) = 2\left(\frac{(L/2)^3}{3E(Wh^3/12)}\right) = \frac{L^3}{EWh^3}$$

$$= 4.2\,\mu m/N$$

Stretching Contribution



$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}}y$$

≈ tanθ

[assume small displacement]

stretching stiffness

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r Wh} = 1.14\,\mu m/\mu N$$

To get the total spring constant
↳ add the bending stiffness to the stretching:

$$k = 4(k_b+k_{st}) = 4(0.24+0.88) = 4.5\,\mu N/\mu m$$

Now, get resonance freq.:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\,N/m}{162\times10^{-12}\,kg}} = 26.5\,kHz$$

ADXL-50 Data Sheet: $f_o = 24\,kHz$ difference?

↳ Capacitive transducer
↘ electrical stiffness