Lecture 21: Capacitive Transducers

- Announcements:
- HW#6 is due Tuesday, Nov. 22
- First project slide due 11/11/11 (email it)
  - Subject & 3 key references
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- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ------------------------------------
- Last Time: Energy Conserving Transducers
- Backtrack a bit and look at last part of Module 11 on micromechanical circuits

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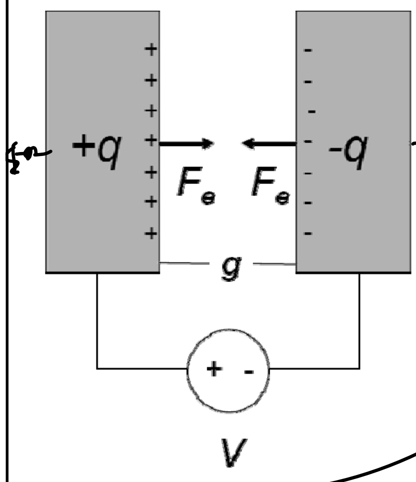
Basic Physics of Electrostatic Actuation



Goal: Determine gap spacing g as a function of input variables

Note: Assume the plates are supported elastically.

1st: Determine the energy of the system.

2nd: Ask, What can I do to Δ the energy of the system?

① change the charge q

② change the separation g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

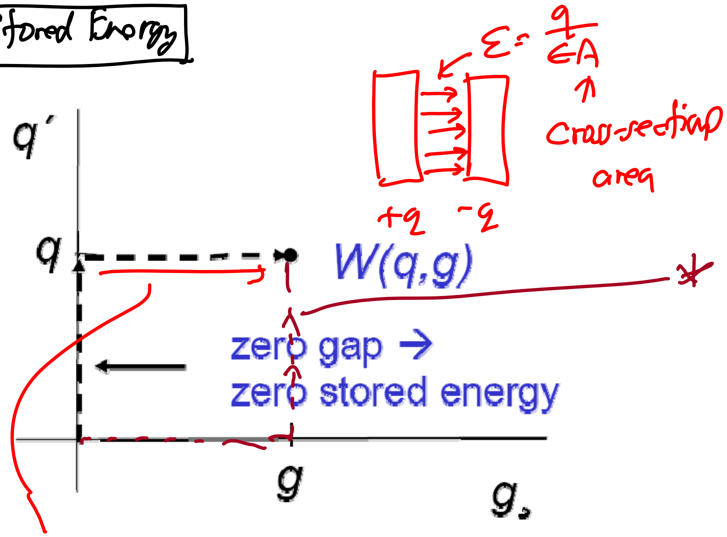
$$dW = Vdq + F_e dg$$

hold q = const. $\rightarrow Vdq \rightarrow 0$

$$dW = F_e dg$$

$$F_e = \left.\frac{dW}{dg}\right|_{q=\text{const.}}$$

Stored Energy



No change in charge: $dq = 0$

$$W = 0 + \int_0^g F_e \, dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A} \quad \text{(independent of } g\text{)}$$

$$\therefore W = \int_0^g F_e \, dg' = F_e g' \Big|_0^g = F_e g$$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$

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* — Work done to charge a C to q at fixed gap g.

$$dW = V\,dq + F_e \, dg \quad (dg = 0)$$

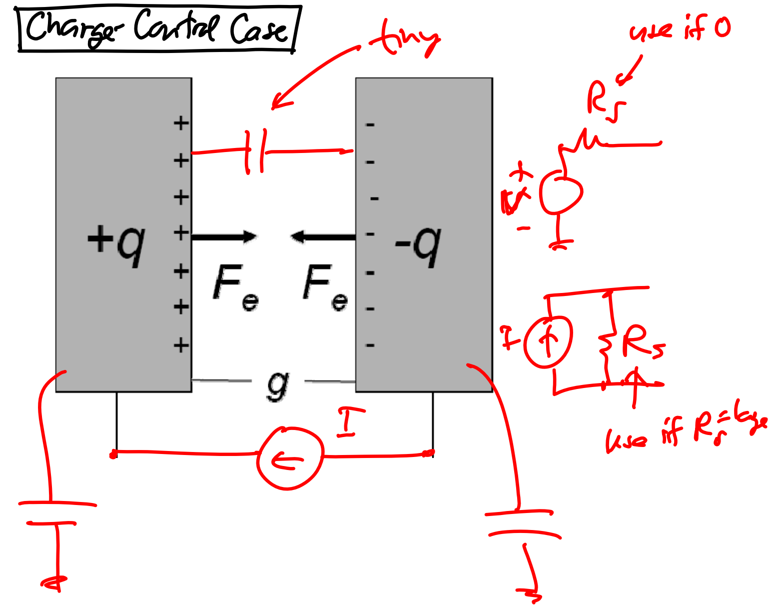
For a capacitor:

$$q = CV \rightarrow V = \frac{q}{C}$$

$$\therefore W(q) = \int_0^q V\,dq = \int_0^q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{q^2}{C}$$

** $= \boxed{\frac{1}{2}\frac{q^2}{\epsilon A} g = W(q)}$

Charge Control Case



From $dW = V\,dq + F_e\,dg$

$\Rightarrow$ Force is given by:

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_{q=\text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}} \rightarrow \text{indep. of gap spacing!}$$

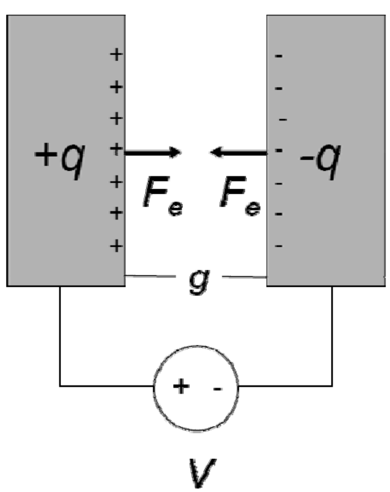
$\Rightarrow$ voltage is given by:

$$V = \left.\frac{\partial W(q,g)}{\partial q}\right|_{g=\text{const.}} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

$$= \frac{qg}{\epsilon A} \therefore \boxed{V = \frac{q}{C}}$$

(consistent w/ what we already know)

Voltage Control



Want to write $F_e = f(V)$

We know this:

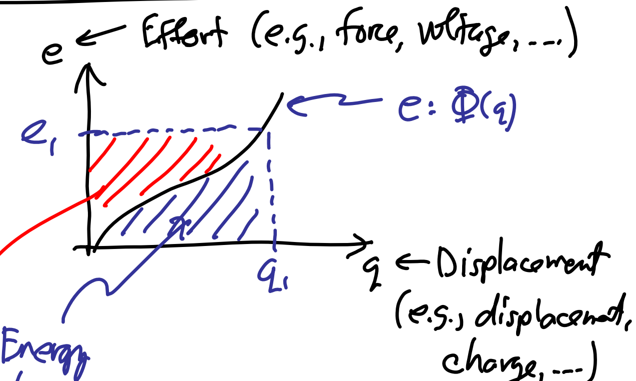
$dW = V\,dq + F_e\,dg$

$W = W(q,g)$

Need: $W'(V,g)$ ← replace charge q w/ V

Can get this using a Legendre transformation.

Energy & Co-Energy



Energy

$$W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-Energy:

$$W'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

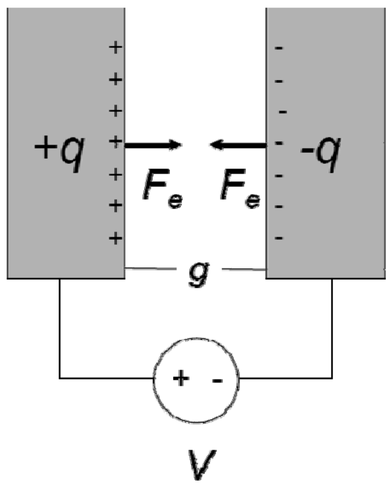
For a linear system, these will be equal.

Can define co-energy as:

$$W'(e) = eq - W(q) \quad \text{(from the plot)} \quad *$$

co-energy ($W'(e)$), energy ($W(q)$)

Co-Energy Formulation for Voltage-Control



$*$ — $W'(V,g) = Vq - W(q,g)$

Differentially, this becomes

$dW'(V,g) = (q\,dV + \cancel{V\,dq}) - dW(q,g)$

$[dW(q,g) = F_e\,dg + \cancel{V\,dq}]$

$\boxed{dW'(V,g) = q\,dV - F_e\,dg}$ ← working Co-Energy expression

Find co-energy in terms of voltage, V:

$$W' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right)V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \checkmark$$ (as expected)

Electrostatic (or Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=const}$$

$$= -\frac{1}{2}\left(\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

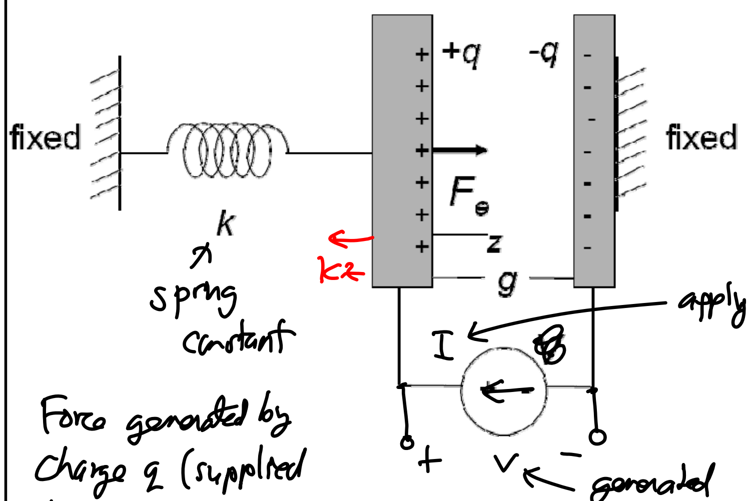
depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=const} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge-Control of a Spring Suspended C



Force generated by charge q (supplied by current I):

$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ ← equilibrium

The gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

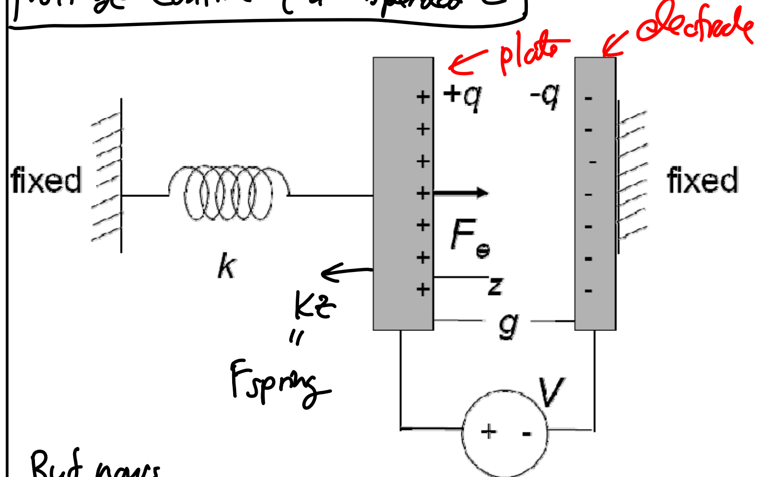
g_0 ← initial gap

q↑ can drive g → 0 in a continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A} g = \frac{q}{\epsilon A}\left(g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V$$

→ V↓ as g↓

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\Big|_q \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

g_0 ← initial gap spacing

g shows up on both sides!

If V↑ → g↓ → F_e↑ (+) Feedback!

If loop gain > 1, then this will go unstable!

↓

plate will collapse! (into the electrode)