Lecture 22: Electrical Stiffness

- **Announcements:**
- First project slide due 11/11/11 (email it)
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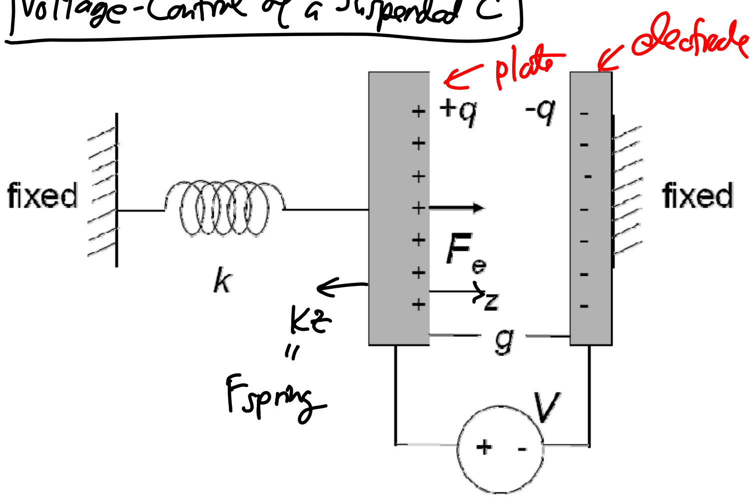
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ------------------------------------
- **Last Time:**

spring

$\boxed{\text{Voltage-Control of a Suspended C}}$



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\bigg|_q \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse! (into the electrode)

Charge: (for a stable gap)

$$q = \frac{\partial W'(V,g)}{\partial V}\bigg|_g = CV \quad \checkmark$$

(as expected)

Stability Analysis

⇒ determine under what conditions voltage-control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when I change g by a small increment dg?

↳ get an increment in the net attractive force F_{net}

$$\underset{(-)}{dF_{net}} = \frac{\partial F_{net}}{\partial g} dg = \underbrace{\left[-\frac{\epsilon A V^2}{g^3} + k \right]} \underset{(-)}{dg}$$

If g↓ → dg = (−), then for stability need F_{net}↓ → dF_{net} = (−)

This needs to be (+)! → otherwise the plates collapse

Thus:

$$\boxed{k > \frac{\epsilon A V^2}{g^3}}$$ (for a stable uncollapsed system)

Pull-in Voltage & Pull-in Gap

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

Substitute (1) into (2):

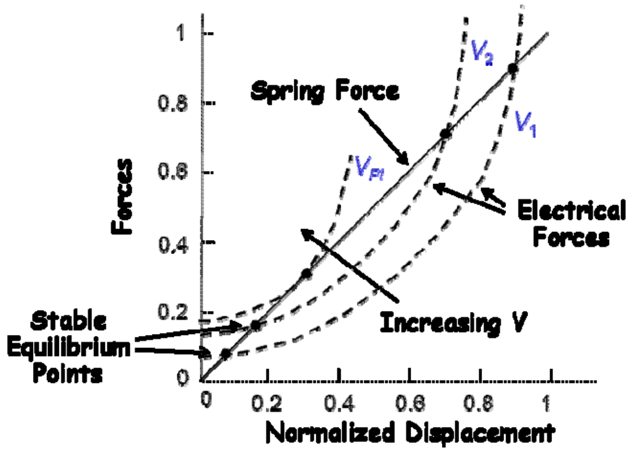
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2 g_{PI}^{\cancel{2}}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_0 - g_{PI})$$

$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_0}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27} \frac{k g_0^3}{\epsilon A}}}$$



Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement

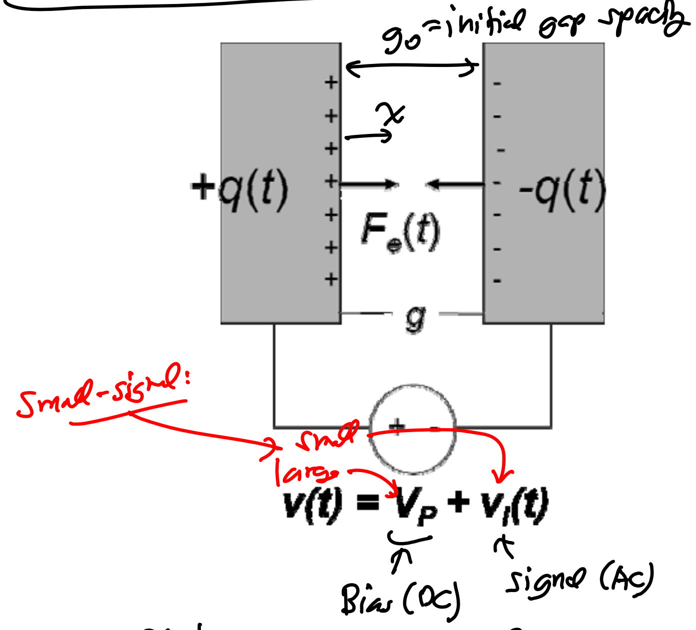
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

- Go through variable naming convention in slide 21 of Lecture Module 12

Linearizing the Voltage-to-Force Transfer Fcn.



$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P \gg v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC Drive Signal}}$$

$$C_0 = \frac{\epsilon A}{g_0} \searrow C(x) = \frac{\epsilon A}{g_0 - x} = C_0\left(1 - \frac{x}{g_0}\right)^{-1}$$

$$[x \ll g_0] \Rightarrow \approx C_0\left(1 + \frac{x}{g_0}\right)$$

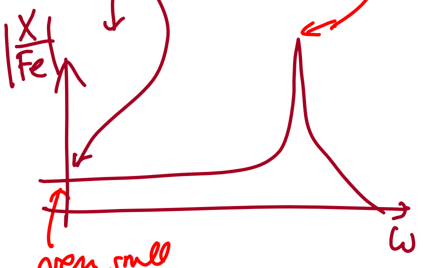
$$\Downarrow \therefore \frac{\partial C}{\partial x} = \frac{C_0}{g_0} = \frac{\epsilon A}{g_0^2}$$

$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_0}{g_0}V_P^2}_{\text{DC offset}} + \boxed{V_P\frac{C_0}{g_0}v_i(t)}$$

~ constant for small amplitudes

∴ this is a linear dependence!

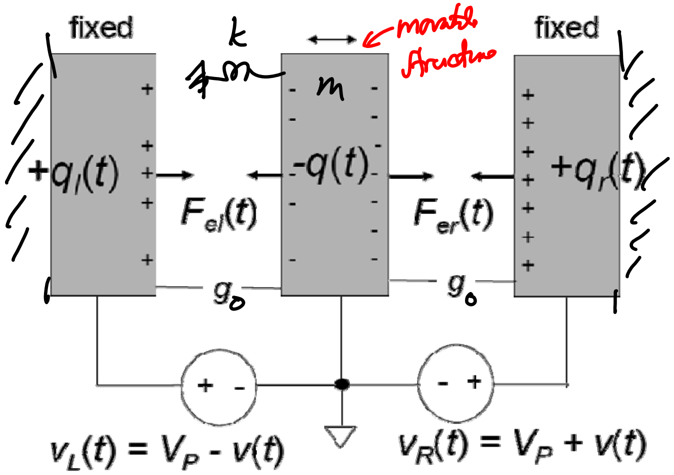
only holds for small amplitudes.



very small response

But still must worry about V_P!

Can Cancel the DC Offset Via Differential Symmetry



$$F_{net}(t) = F_{er}(t) - F_{el}(t)$$

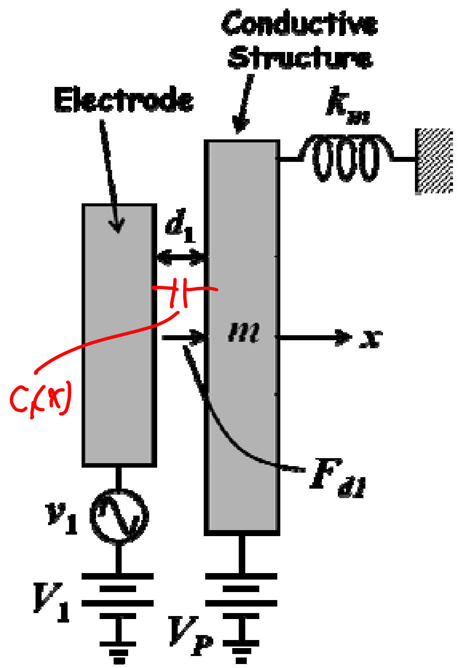
$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

$$\therefore \boxed{F_{net}(t) = 2V_P\frac{\partial C}{\partial x}v(t) = 2V_P\frac{C_0}{g_0}v(t)}$$

↳ Linear w/ $v(t)$!

Nonlinearity Still Effects Us



More Complete Expression

$$C_1(x) = \frac{\epsilon A}{d_1 + x} = C_{o1}\left(1+\frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1+\frac{x}{d_1}\right)^{-2}$$

[Expand into Taylor series]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, ...

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}(V_P - V_1 - v_1)^2 = \frac{1}{2}\frac{\partial C}{\partial x}(\underset{\uparrow}{V_{P1}} - v_1)^2$$

$V_{P1} = V_P - V_1$

[small displacement: $x \ll d_1$]

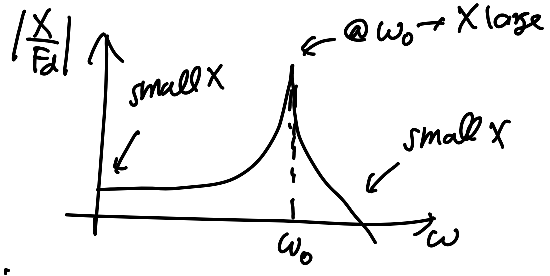
$\frac{1}{2}(1+\cos 2\omega_o t)$

$$F_{d1} = \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)(1+A_1 x)(V_{P1}^2 - 2V_{P1}v_1 + \underset{\uparrow}{v_1^2})$$

$\cos^2\omega_o t$

$$= \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)\left\{V_{P1}^2 - 2\underline{V_{P1}v_1} + v_1^2 + \underline{A_1 V_{P1}^2 x} - 2A_1 V_{P1} x v_1 + A_1 x v_1^2\right\}$$

Resonance:



@ resonance:

$$x = \frac{QF_{d1}}{\underset{\uparrow}{j}k} = \frac{Q}{jk}\frac{\partial C}{\partial x}V_{P1}v_1$$

90° phase shift

$$v_1 = |v_1|\cos\omega_o t \underset{\uparrow}{\longrightarrow} x = |x|\sin\omega_o t$$

90° phase-shifted

Force terms @ ω_0

$$F_{d1}\big|_{\omega_0} = \underbrace{V_{P1}\frac{C_{o1}}{d_1}W_1|\cos\omega_0 t}_{\text{drive force term}} + \overbrace{V_{P1}^2\frac{C_{o1}}{d_1^2}}^{k_e \to \text{electrical stiffness}}|X|\sin\omega_0 t$$

proportional to x

$90°$ phase shifted f/

$\therefore$ in phase as displacement!

$\therefore$ it's a stiffness!

Electrical Stiffness:

① A negative spring constant!

② Depends on V_P!

$$\boxed{k_e = V_{P1}^2\frac{C_{o1}}{d_1^2} = V_{P1}^2\frac{\epsilon A}{d_1^3}}$$

What does this do for us?

$\to$ it affects resonance freq.!

$$\omega_0 = \sqrt{\frac{k_m}{m}}$$

k_m ← mechanical spring const.

m ← mass

got this w/o V_P

Apply V_P: $\omega_0' = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_m - k_e}{m}} = \sqrt{\frac{k_m}{m}}\left(1-\frac{k_e}{k_m}\right)^{1/2}$

$$\boxed{\omega_0' = \omega_0\left(1-\frac{V_{P1}^2}{k_m}\frac{\epsilon A}{d_1^3}\right)^{1/2}}$$