Lecture 22: Electrical Stiffness

- Announcements:
- First project slide due 11/9/11 (email it)
  - Subject & 3 key references
- No lecture Tuesday, next week, Nov. 13
  - As before, we will make it up by going longer in some lectures, starting today
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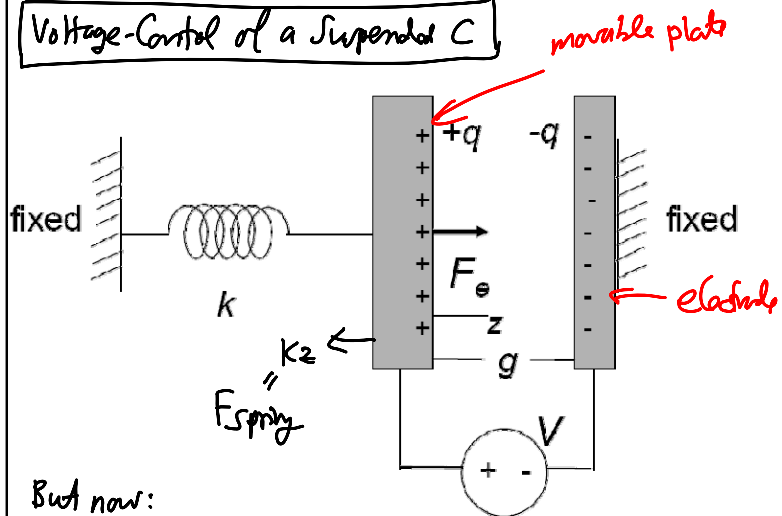
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ----------------------------------
- Last Time:
- Voltage-control of a spring-suspended C
- Get (+) feedback loop → instability
- We we're in the middle of analyzing that instability

over

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\Big|_q \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1 ⇒ this will go unstable!

plates will collapse (catastrophic!)

Charge: (for a stable gap)

$$q = \frac{\partial W'(V,g)}{\partial V}\Big|_g = CV \checkmark$$

Stability Analysis

⇒ determine under what conditions voltage-control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when g changes by increment dg?
↘ get an increment in the net attractive force dF_{net}

$$\underset{(-)}{dF_{net}} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right] \underset{(-)}{dg}$$

If $g\downarrow \rightarrow dg = (-)$ and
For stability, need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

then need this to be (+) → otherwise plates will collapse!

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed state)

Pull-in Voltage & Pull-In Gap

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

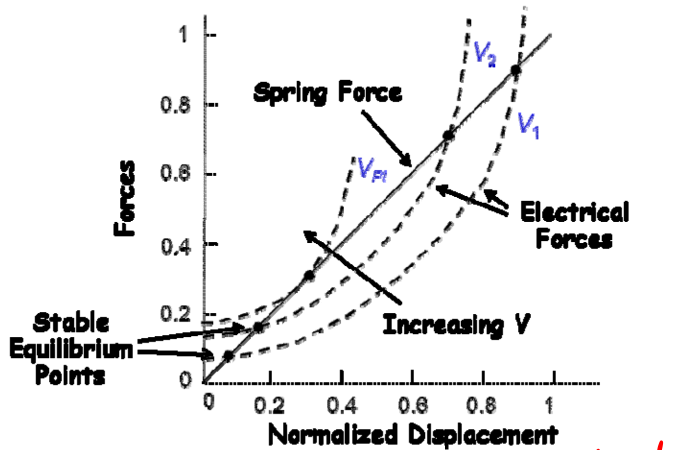
Substitute (1) into (2):

$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2\cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_0 - g_{PI})$$

$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI} \quad \therefore \boxed{g_{PI} = \frac{2}{3} g_0}$$

When the gap is driven by a voltage to (2/3) the initial gap → Collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow V_{PI} = \sqrt{\frac{8}{27}\frac{k g_0^3}{\epsilon A}}$$



Capacitive-gap transducers

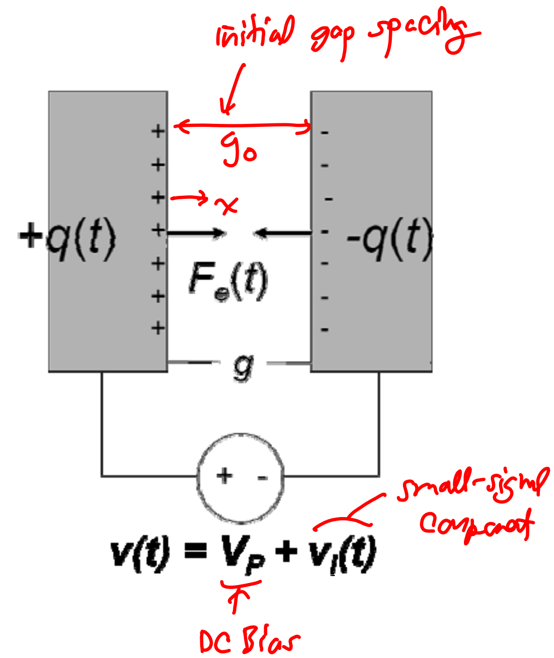
Advantages of ~~Electrostatic~~ Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale
- Go through variable naming convention in slide 21 of Lecture Module 12

Linearizing the Voltage-to-Force Transfer Function



$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P \gg v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC Drive Signal}}$$

$$C_0 = \frac{\epsilon A}{g_0} \rightarrow C(x) = \frac{\epsilon A}{g_0 - x} = C_0\left(1 - \frac{x}{g_0}\right)^{-1}$$

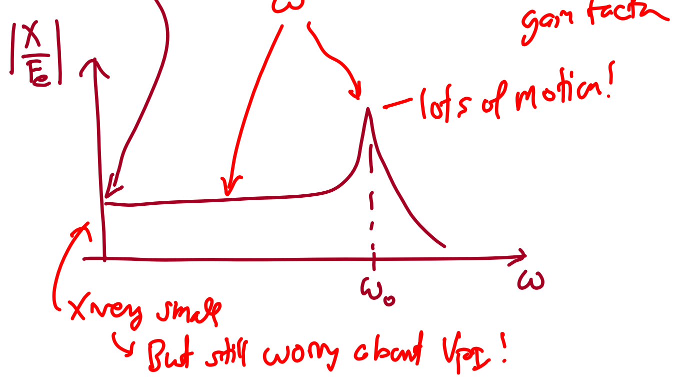
$$[x \ll g_0] \Rightarrow \simeq C_0\left(1 + \frac{x}{g_0}\right)$$

$$\therefore \frac{\partial C}{\partial x} = \frac{C_0}{g_0} = \frac{\epsilon A}{g_0^2}$$

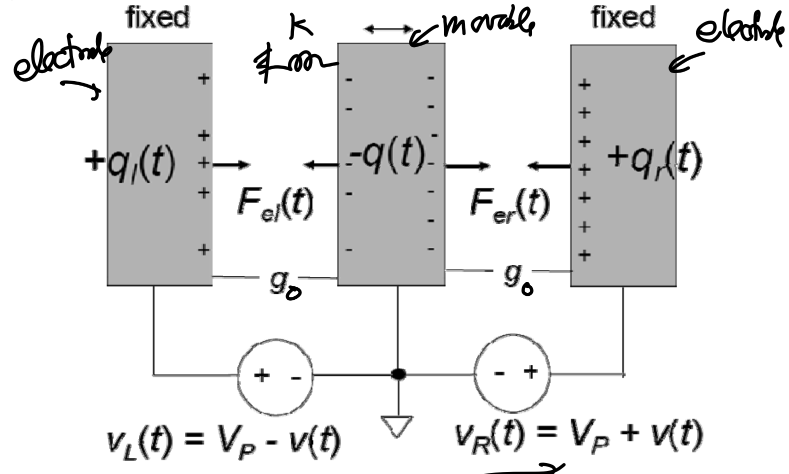
$$F_e(t) = \underbrace{\frac{1}{2}\frac{C_0}{g_0}V_P^2}_{\text{DC Offset}} + \underbrace{\overbrace{V_P\frac{C_0}{g_0}}^{k}v_i(t)}_{\omega}$$

$k v_i(t)$

signal force $\sim V_P$
↑ gain factor



Cancel the DC Offset Using Differential Symmetry



differential input into balanced structure

$$F_{net} = F_{er}(t) - F_{el}(t)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

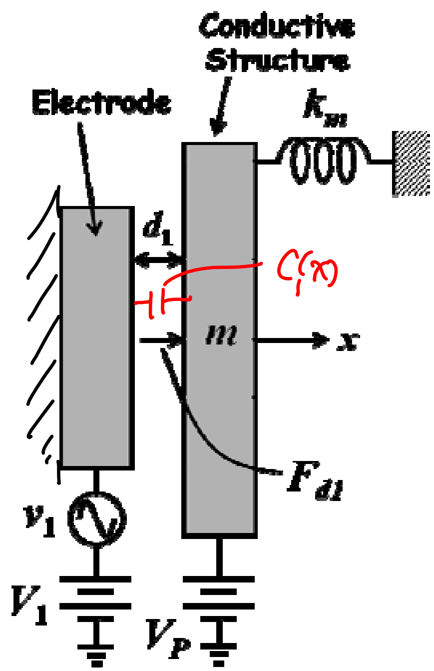
$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

$$\therefore \boxed{F_{net}(t) = 2V_P\frac{\partial C}{\partial x}v(t) = 2V_P\frac{C_0}{g_0}v(t)}$$

→ No DC component → less pull-in problems, but only to the extent of matching

→ Linear w/ $v(t)$!

Nonlinearity Still Affects Us!



More Complete Expression!

$$C_1(x) = \frac{\epsilon A}{d_1 + x} = C_{o1}\left(1+\frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1+\frac{x}{d_1}\right)^{-2}$$

[Expand into Taylor series]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, …

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}\left(V_P - V_1 - v_1\right)^2 = \frac{1}{2}\frac{\partial C}{\partial x}\left(V_{P1} - v_1\right)^2$$

$V_{P1} = V_P - V_1$

[small displacements: $x \ll d_1$]

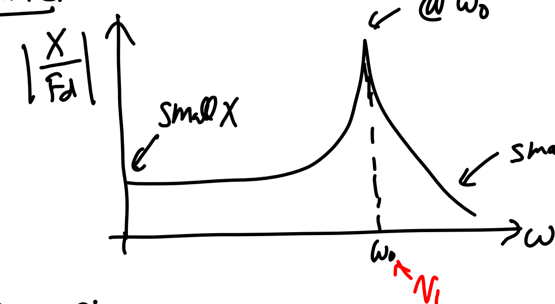
$$F_{d1} = \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)\left(1 + A_1 x\right)\left(V_{P1}^2 - 2V_{P1}v_1 + v_1^2\right)$$

$\frac{|v_1|^2}{2}(1+\cos 2\omega_o t)$ ω_o

$= (|v_1|\cos\omega_o t)^2$

$$= \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)\left\{V_{P1}^2 - 2V_{P1}v_1 + v_1^2 + A_1 V_{P1}^2 x - 2A_1 V_{P1} x v_1 + A_1 x v_1^2\right\}$$

Resonance:



@ resonance:

$$x = \frac{QF_{d1}}{jk} = \frac{Q}{jk}\frac{\partial C_1}{\partial x}V_{P1}v_1$$ ← x in terms of v_1

90° phase shift

$v_1 = |v_1|\cos\omega_o t \longrightarrow x = |X|\sin\omega_o t$

90° phase shift

Force terms @ ω_o

$k_e \rightarrow$ electrical stiffness

$$F_{d1}\Big|_{\omega_o} = \underbrace{V_{P1}\frac{C_{o1}}{d_1}|v_i|\cos\omega_o t}_{\text{drive force term (from } v_i)} + \overbrace{V_{P1}^2\frac{C_{o1}}{d_1^2}}^{k_e}|x|\sin\omega_o t$$

drive force term (from v_i)

proportional to x!

$90°$ phase-shifted from

$\therefore$ in phase w/ displacement!

$\therefore$ it's a stiffness!

Electrical Stiffness:

① A negative spring constant!

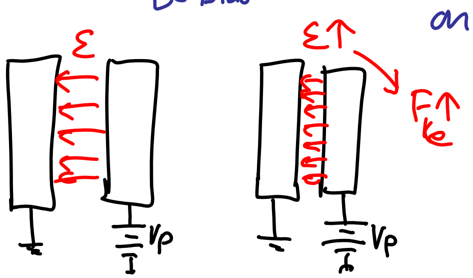
② Dependent on V_P:

$$\boxed{k_e = V_{P1}^2\frac{C_{o1}}{d_1^2} = V_{P1}^2\frac{\epsilon A}{d_1^3}}$$

overlap area of the C_{o1}

DC Bias

3^{rd} power dependence on gap!



$k_e \rightarrow$ can affect resonance freq., f_o!

$\omega_o \triangleq$ radian resonance freq. w/ no V_P applied (i.e., $V_{P1} = 0V$)

$$\omega_o = \sqrt{\frac{k_m}{m}}$$

$k_m \leftarrow$ mech. stiffness

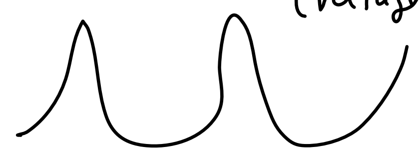
$m \leftarrow$ mass

$$\omega_o' = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_m - k_e}{m}} = \sqrt{\frac{k_m}{m}}\left(1 - \frac{k_e}{k_m}\right)^{1/2}$$

system stiffness

$$\boxed{\omega_o' = \omega_o\left[1 - \frac{V_{P1}^2}{k_m}\frac{\epsilon A}{d_1^3}\right]^{1/2}}$$

Now a fcn of DC Bias, V_{P1}! (voltage-controllable!)



- Go through Module 12 slides 26-35