CUSUM, MA and EWMA Control Charts

Increasing the sensitivity and getting ready for automated control:

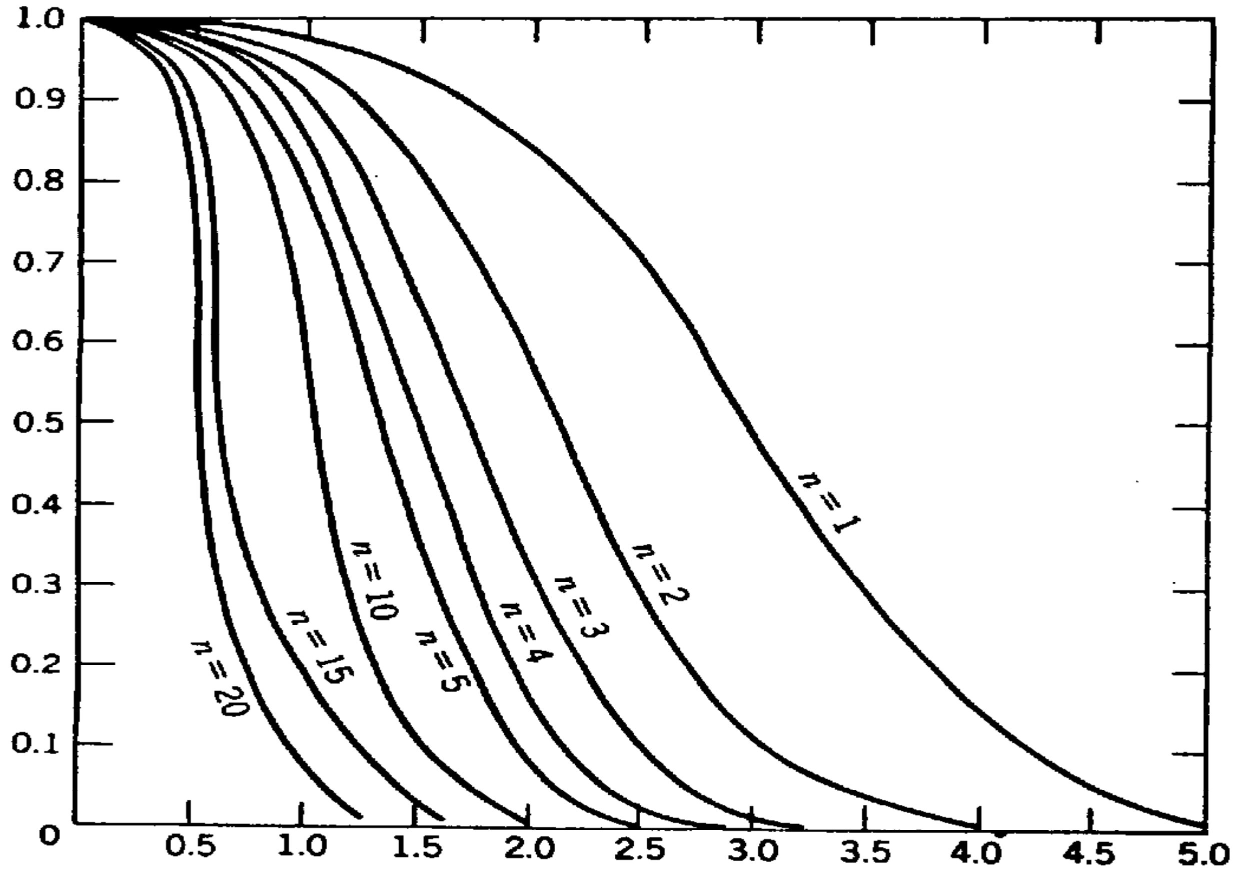
The Cumulative Sum chart, the Moving Average and the Exponentially Weighted Moving Average Charts.

Shewhart Charts cannot detect small shifts

The charts discussed so far are variations of the *Shewhart* chart: each new point depends only on one subgroup.

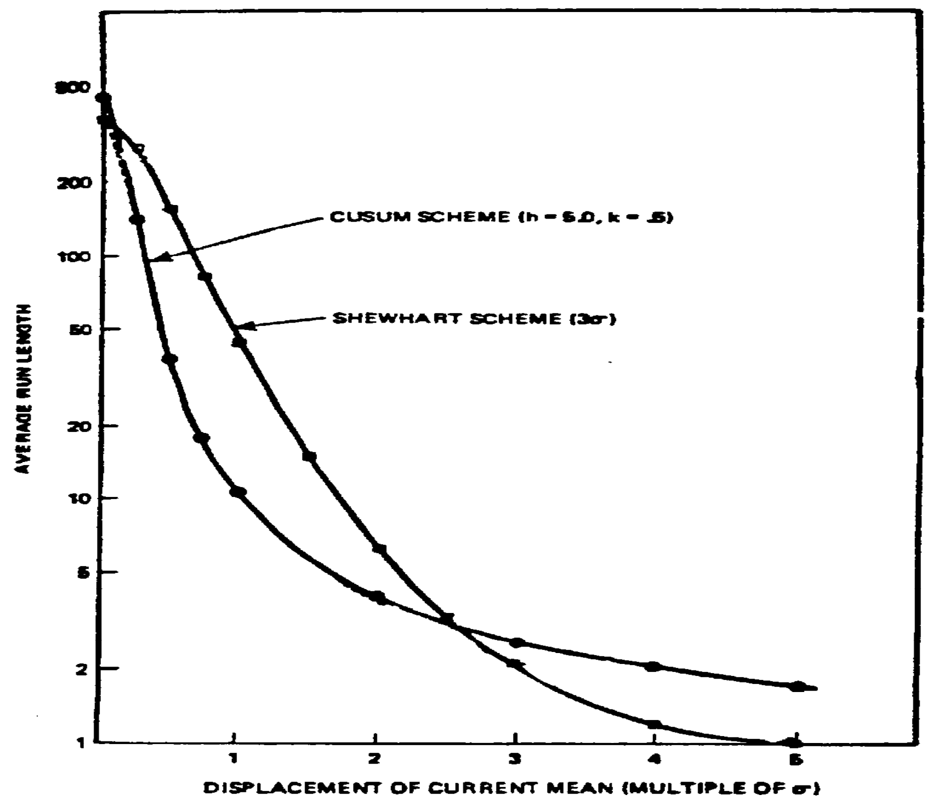
Shewhart charts are sensitive to large process shifts.

The probability of detecting small shifts fast is rather small:



Cumulative-Sum Chart

If each point on the chart is the *cumulative history* (integral) of the process, systematic shifts are easily detected. Large, abrupt shifts are not detected as fast as in a Shewhart chart.



CUSUM charts are built on the principle of *Maximum Likelihood Estimation* (MLE).

Maximum Likelihood Estimation

The "correct" choice of probability density function (pdf) moments maximizes the collective likelihood of the observations.

If x is distributed with a pdf(x,θ) with unknown θ, then θ can be estimated by solving the problem:

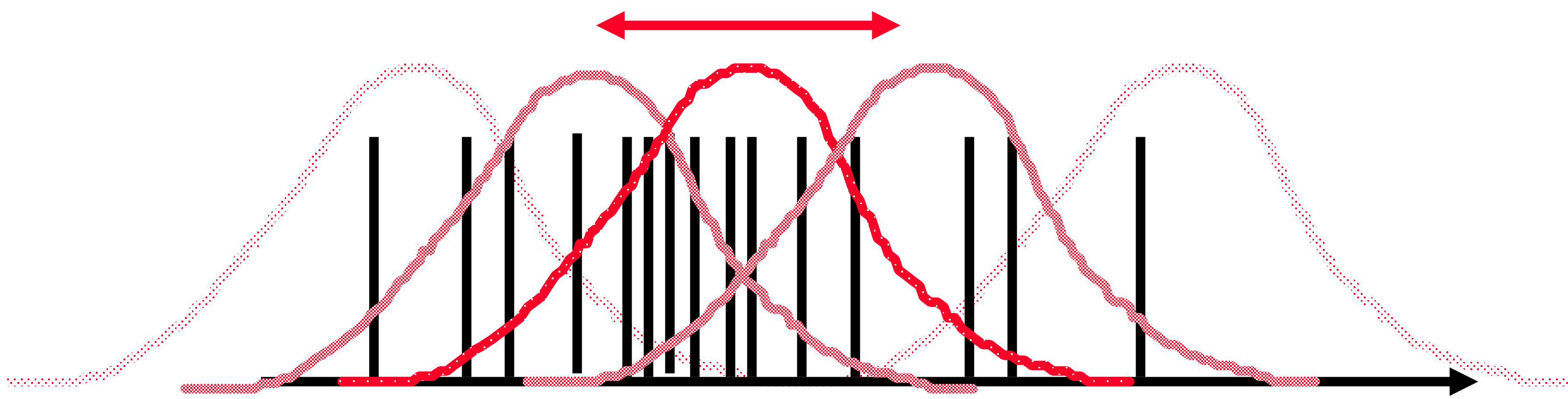
$$\max_{\theta}\left[\prod_{i=1}^{m} \mathrm{pdf}(x_i,\theta)\right]$$

This concept is good for estimation as well as for comparison.

Maximum Likelihood Estimation Example

To estimate the mean value of a normal distribution, collect the observations $x_1, x_2, \ldots, x_m$ and solve the non-linear programming problem:

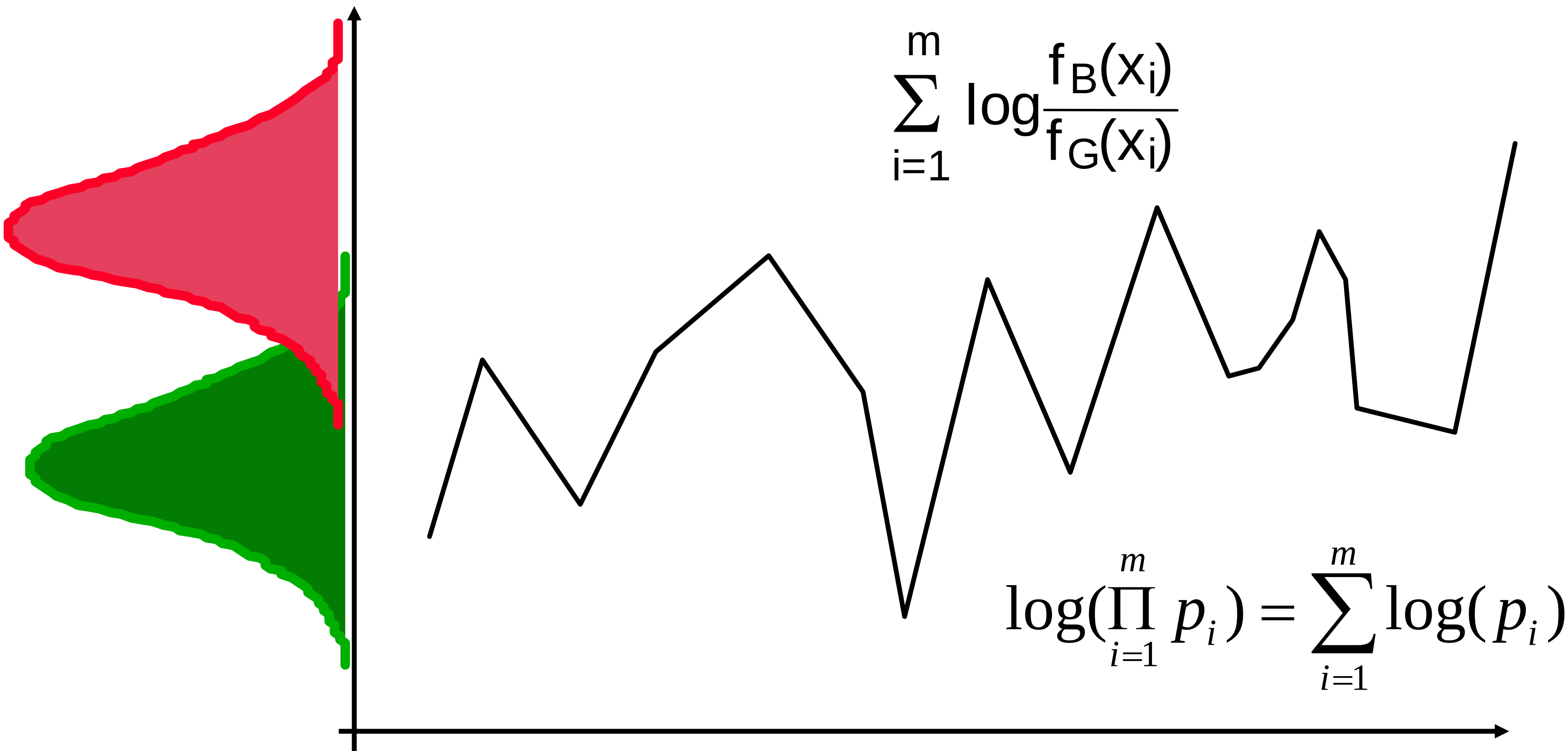
$$\max_{\hat{\mu}} \left\{ \prod_{i=1}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\hat{\mu}}{\sigma}\right)^2} \right\} \quad or \quad \min_{\hat{\mu}} \left\{ \sum_{i=1}^{m} \log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\hat{\mu}}{\sigma}\right)^2} \right) \right\}$$



MLE Control Schemes

If a process can have a "good" or a "bad" state (with the control variable distributed with a pdf f_G or f_B respectively).

This statistic will be small when the process is "good" and large when "bad":



MLE Control Schemes (cont.)

Note that this counts from the beginning of the process. We choose the best k points as "calibration" and we get:

$$S_m = \sum_{i=1}^{m} \log\frac{f_B(x_i)}{f_G(x_i)} - \min_{k<m} \sum_{i=1}^{k} \log\frac{f_B(x_i)}{f_G(x_i)} > L$$

or

$$S_m = \max\left(S_{m-1} + \log\frac{f_B(x_m)}{f_G(x_m)}, 0\right) > L$$

This way, the statistic S_m keeps a cumulative score of all the "bad" points. Notice that we need to know what the "bad" process is!

The Cumulative Sum chart

If θ is a mean value of a normal distribution, is simplified to:

$$S_m = \sum_{i=1}^{m} (\bar{x}_i - \mu_0)$$

where μ_0 is the target mean of the process. This can be monitored with V-shaped or tabular “limits”.

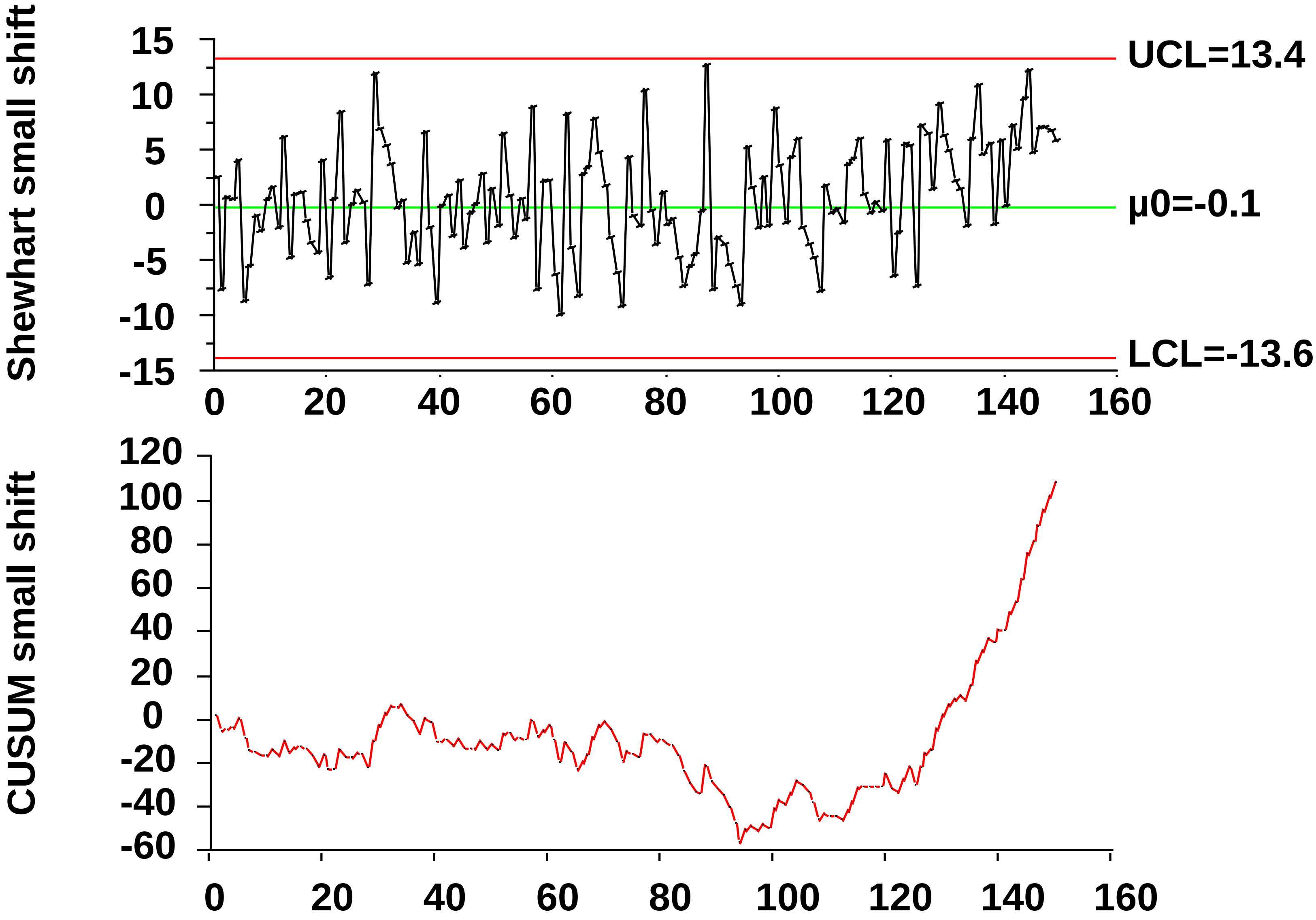
<u>Advantages</u>

The CUSUM chart is very effective for small shifts and when the subgroup size n=1.

<u>Disadvantages</u>

The CUSUM is relatively slow to respond to large shifts. Also, special patterns are hard to see and analyze.

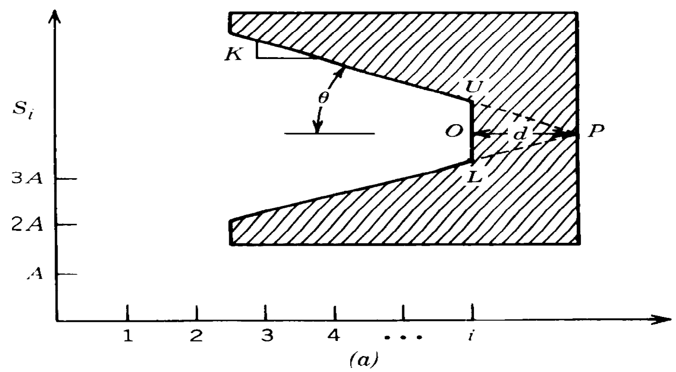
Example

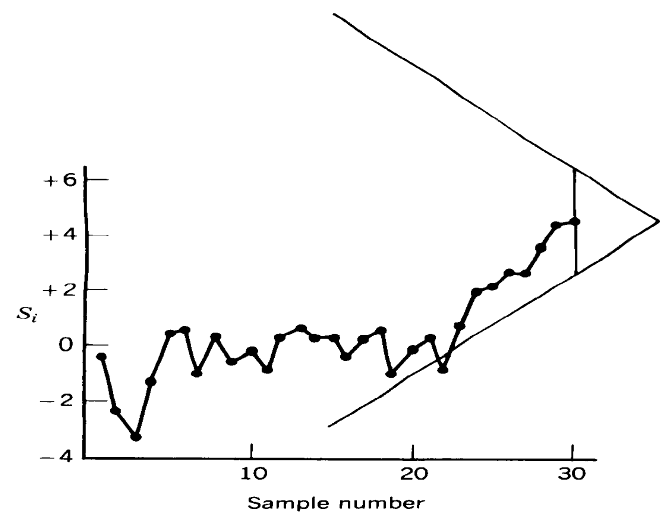

Shewhart small shift
15
10
5
0
-5
-10
-15
UCL=13.4
μ0=-0.1
LCL=-13.6
0
20
40
60
80
100
120
140
160
CUSUM small shift
120
100
80
60
40
20
0
-20
-40
-60
0
20
40
60
80
100
120
140
160

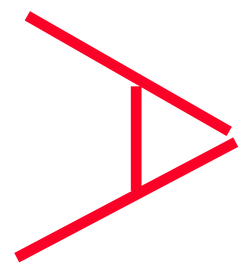
The V-Mask CUSUM design

for standardized observations $y_i=(x_i-\mu_o)/\sigma$

Need to set L(0) (i.e. the run length when the process is in control), and L(δ) (i.e. the run-length for a *specific* deviation).







The V-Mask CUSUM design

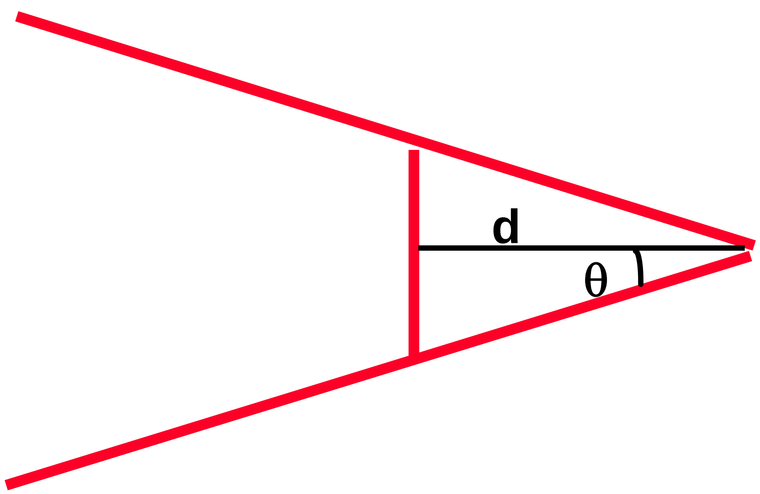
for standardized observations $y_i=(x_i-\mu_o)/\sigma$

δ is the amount of shift (normalized to σ) that we wish to detect with type I error α and type II error β.

A is a scaling factor: it is the horizontal distance between successive points in terms of unit distance on the vertical axis.

$$d = \left(\frac{2}{\delta^2}\right) \ln\left(\frac{1-\beta}{\alpha}\right)$$

$$\theta = \tan^{-1}\left(\frac{\delta}{2A}\right) \qquad \delta = \frac{\Delta}{\sigma_{\overline{x}}}$$



ARL vs. Deviation for V-Mask CUSUM

δ = Deviation from Target Value (in standard deviations)

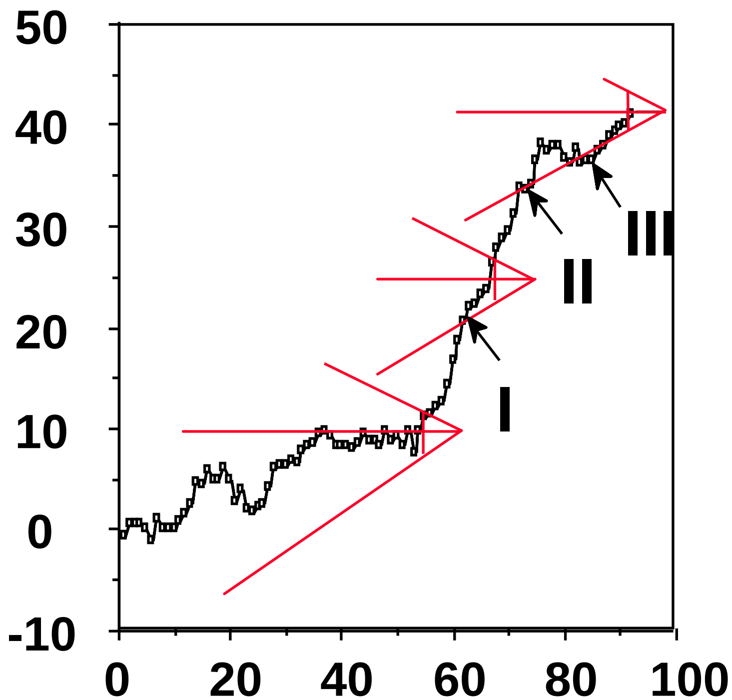
$L(0)$ = Average run length when process is in control.

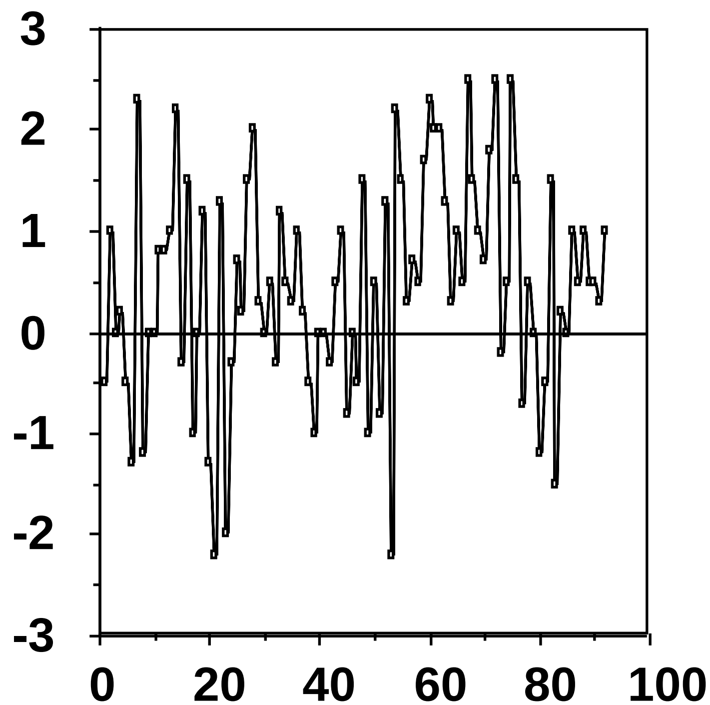
δ		50	100	200	300	400
	$(A/\sigma_{\bar{x}})\tan\theta$	0.125			0.195	
0.25	d	47.6			46.2	
	$L(0.25)$	28.3			74.0	
	$(A/\sigma_{\bar{x}})\tan\theta$	0.25	0.28	0.29	0.28	0.28
0.50	d	17.5	18.2	21.4	24.7	27.3
	$L(0.5)$	15.8	19.0	24.0	26.7	29.0
	$(A/\sigma_{\bar{x}})\tan\theta$	0.375	0.375	0.375	0.375	0.375
0.75	d	9.2	11.3	13.8	15.0	16.2
	$L(0.75)$	8.9	11.0	13.4	14.5	15.7
	$(A/\sigma_{\bar{x}})\tan\theta$	0.50	0.50	0.50	0.50	0.50
1.0	d	5.7	6.9	8.2	9.0	9.6
	$L(1.0)$	6.1	7.4	8.7	9.4	10.0
	$(A/\sigma_{\bar{x}})\tan\theta$	0.75	0.75	0.75	0.75	0.75
1.5	d	2.7	3.3	3.9	4.3	4.5
	$L(1.5)$	3.4	4.0	4.6	5.0	5.2
	$(A/\sigma_{\bar{x}})\tan\theta$	1.0	1.0	1.0	1.0	1.0
2.0	d	1.5	1.9	2.2	2.4	2.5
	$L(2.0)$	2.26	2.63	2.96	3.15	3.3

CUSUM chart of furnace Temperature difference

Detect 2C^o, σ =1.5C^o, (i.e. δ=1.33), α=.0027 β=0.05, A=1

=> θ = 18.43^o, d = 6.6



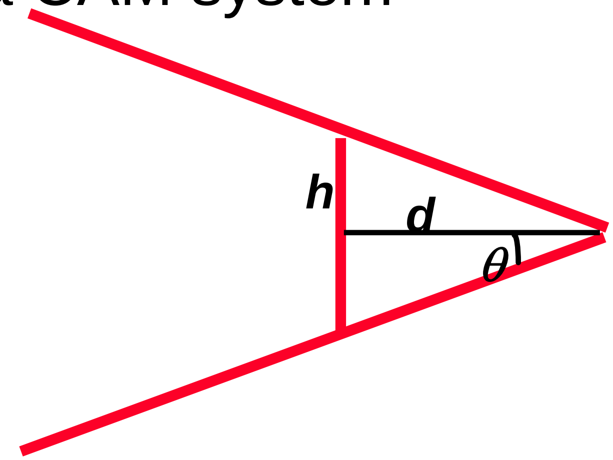


Tabular CUSUM

A tabular form is easier to implement in a CAM system

$C_i^+ = \max [0, \bar{x}_i - (\mu_o + k) + C_{i-1}^+]$

$C_i^- = \max [0, (\mu_o - k) - \bar{x}_i + C_{i-1}^-]$



$C_0^+ = C_0^- = 0$

$k = (\delta/2)/\sigma$

$h = d\sigma_x \tan(\theta)$

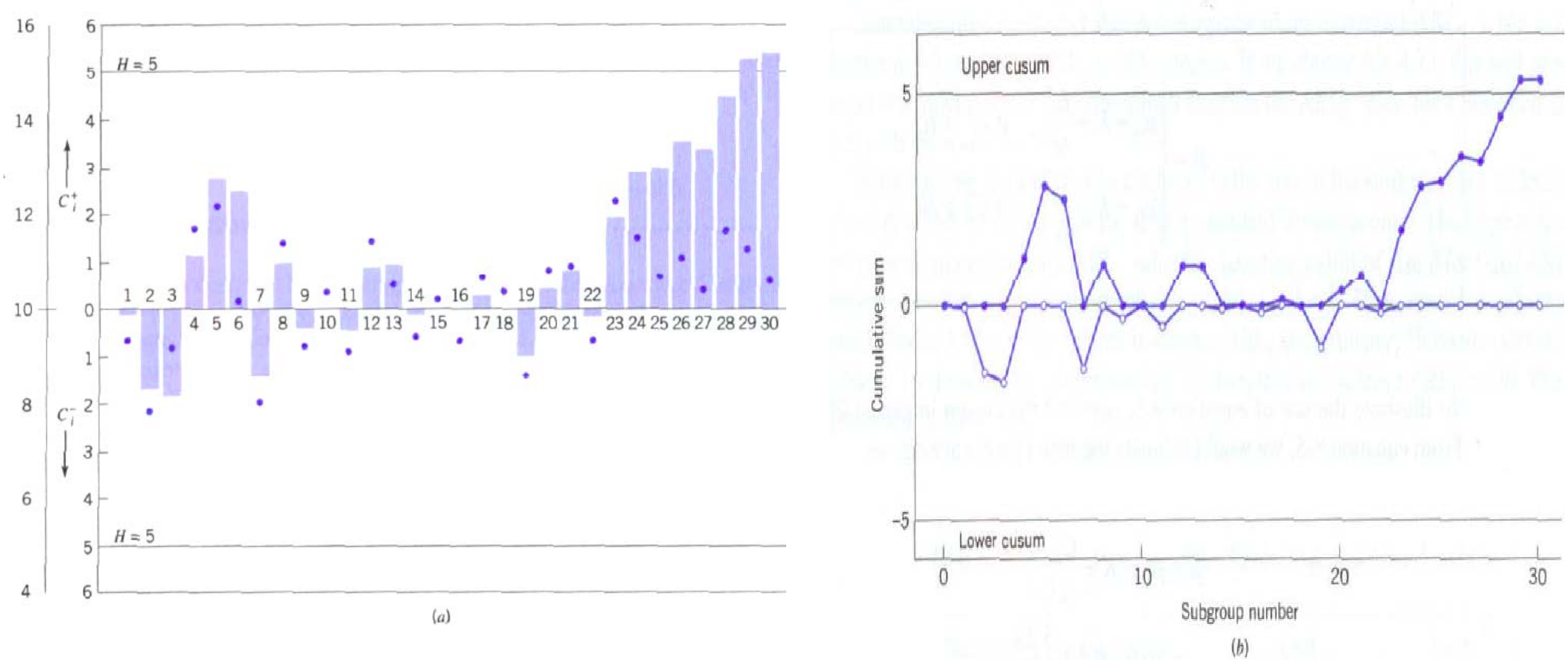
ARL Performance of the Tabular Cusum with $k = \frac{1}{2}$ and $h =$ 4 or $h = 5$

Shift in Mean (multiple of σ)	$h = 4$	$h = 5$
0	168	465
0.25	74.2	139
0.50	26.6	38.0
0.75	13.3	17.0
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

Tabular CUSUM Example

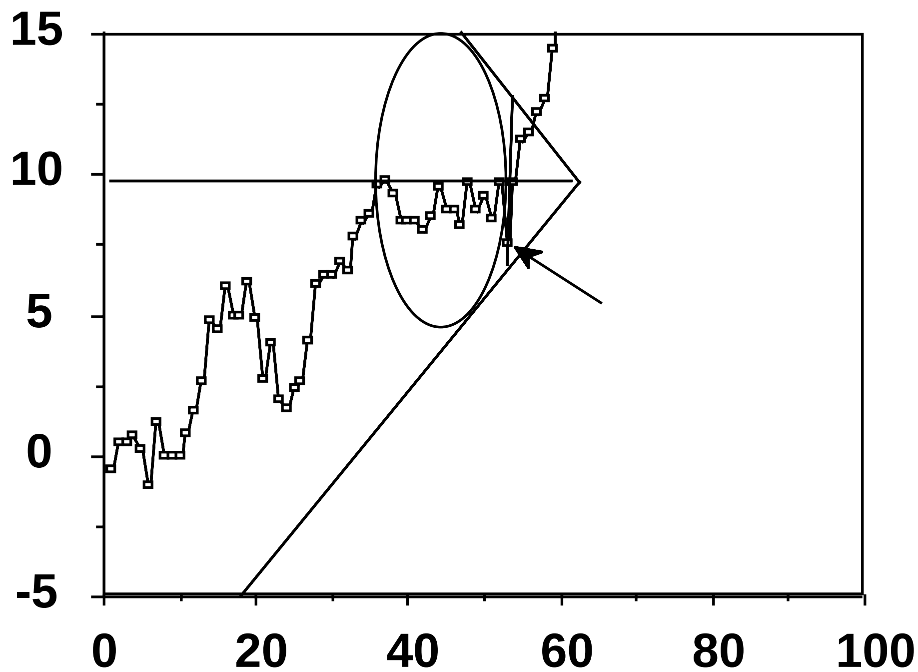
		(a)			(b)		
Period i	x_i	$x_i - 10.5$	C_i^+	N^+	$9.5 - x_i$	C_i^-	N^-
1	9.45	−1.05	0	0	0.05	0.05	1
2	7.99	−2.51	0	0	1.51	1.56	2
3	9.29	−1.21	0	0	0.21	1.77	3
4	11.66	1.16	1.16	1	−2.16	0	0
5	12.16	1.66	2.82	2	−2.66	0	0
6	10.18	−0.32	2.50	3	−0.68	0	0
7	8.04	−2.46	0.04	4	1.46	1.46	1
8	11.46	0.96	1.00	5	−1.96	0	0
9	9.20	−1.3	0	0	0.30	0.30	1
10	10.34	−0.16	0	0	−0.84	0	0
11	9.03	−1.47	0	0	0.47	0.47	1
12	11.47	0.97	0.97	1	−1.97	0	0
13	10.51	0.01	0.98	2	−1.01	0	0
14	9.40	−1.10	0	0	0.10	0.10	1
15	10.08	−0.42	0	0	−0.58	0	0
16	9.37	−1.13	0	0	0.13	0.13	1
17	10.62	0.12	0.12	1	−1.12	0	0
18	10.31	−0.19	0	0	−0.81	0	0
19	8.52	−1.98	0	0	0.98	0.98	1
20	10.84	0.34	0.34	1	−1.34	0	0
21	10.90	0.40	0.74	2	−1.40	0	0
22	9.33	−1.17	0	0	0.17	0.17	1
23	12.29	1.79	1.79	1	−2.79	0	0
24	11.50	1.00	2.79	2	−2.00	0	0
25	10.60	0.10	2.89	3	−1.10	0	0
26	11.08	0.58	3.47	4	−1.58	0	0
27	10.38	−0.12	3.35	5	−0.88	0	0
28	11.62	1.12	4.47	6	−2.12	0	0
29	11.31	0.81	5.28	7	−1.81	0	0
30	10.52	0.02	5.30	8	−1.02	0	0

Various Tabular CUSUM Representations



CUSUM Enhancements

To speed up CUSUM response one can use "modified" V masks:



Other solutions include the application of Fast Initial Response (FIR) CUSUM, or the use of combined CUSUM-Shewhart charts.

General MLE Control Schemes

Since the MLE principle is so general, control schemes can be built to detect:

- single or multivariate deviation in means
- deviation in variances
- deviation in covariances

An important point to remember is that MLE schemes need, implicitly or explicitly, a definition of the "bad" process.

The calculation of the ARL is complex but possible.

Control Charts Based on Weighted Averages

Small shifts can be detected more easily when multiple samples are combined.

Consider the average over a "moving window" that contains w subgroups of size n:

$$M_t = \frac{\bar{x}_t + \bar{x}_{t-1} + \ldots + \bar{x}_{t-w+1}}{w}$$

$$V(M_t) = \frac{\sigma^2}{n\,w}$$

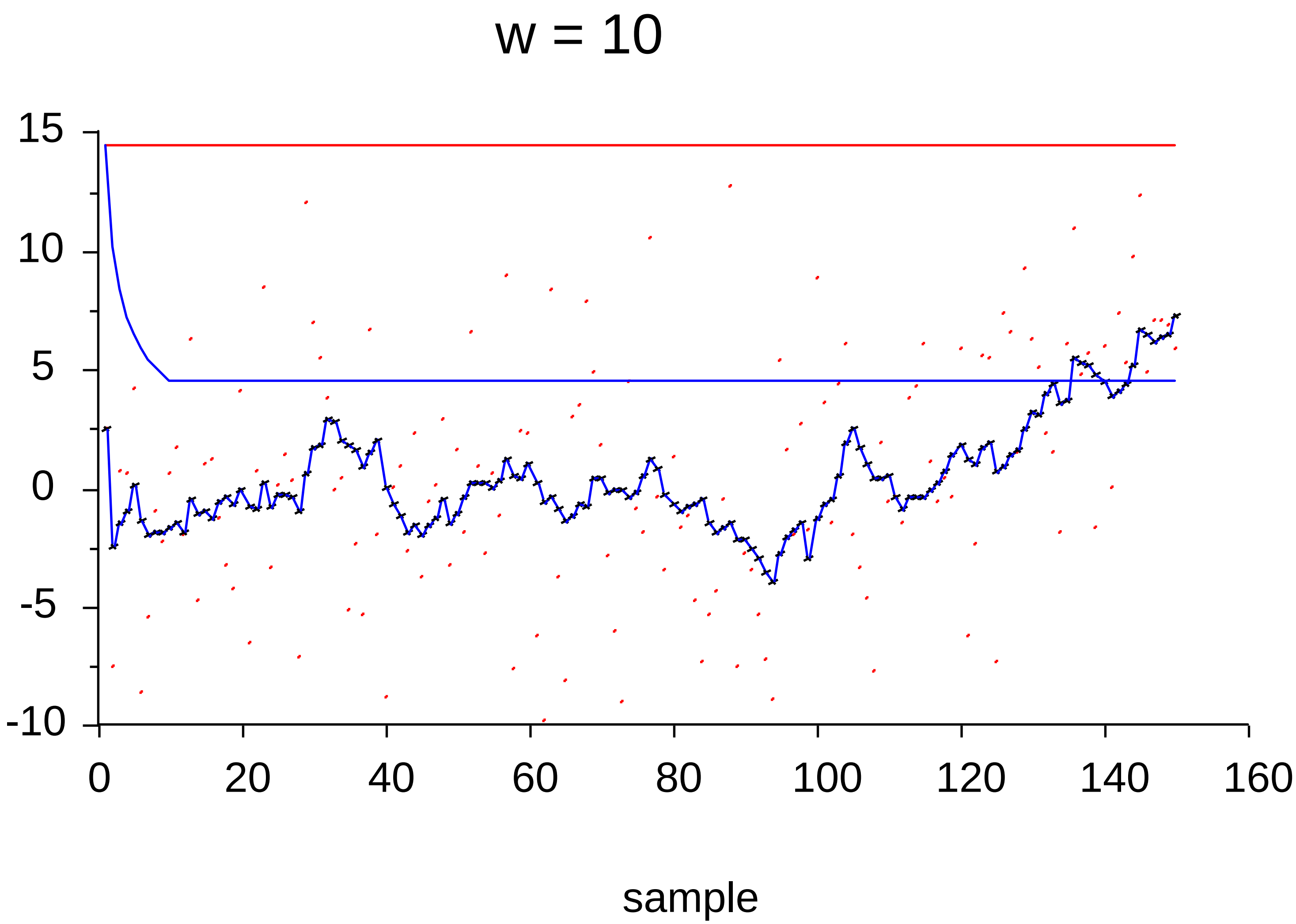
The 3-sigma control limits for M_t are:

$$UCL = \bar{\bar{x}} + \frac{3\,\sigma}{\sqrt{n\,w}}$$

$$LCL = \bar{\bar{x}} - \frac{3\,\sigma}{\sqrt{n\,w}}$$

Limits are wider during start-up and stabilize after the first w groups have been collected.

Example - Moving average chart



The Exponentially Weighted Moving Average

If the CUSUM chart is the sum of the entire process history, maybe a weighed sum of the recent history would be more meaningful:

$$z_t = \lambda \bar{x}_t + (1 - \lambda) z_{t-1} \qquad 0 < \lambda < 1 \quad z_0 = \bar{\bar{x}}$$

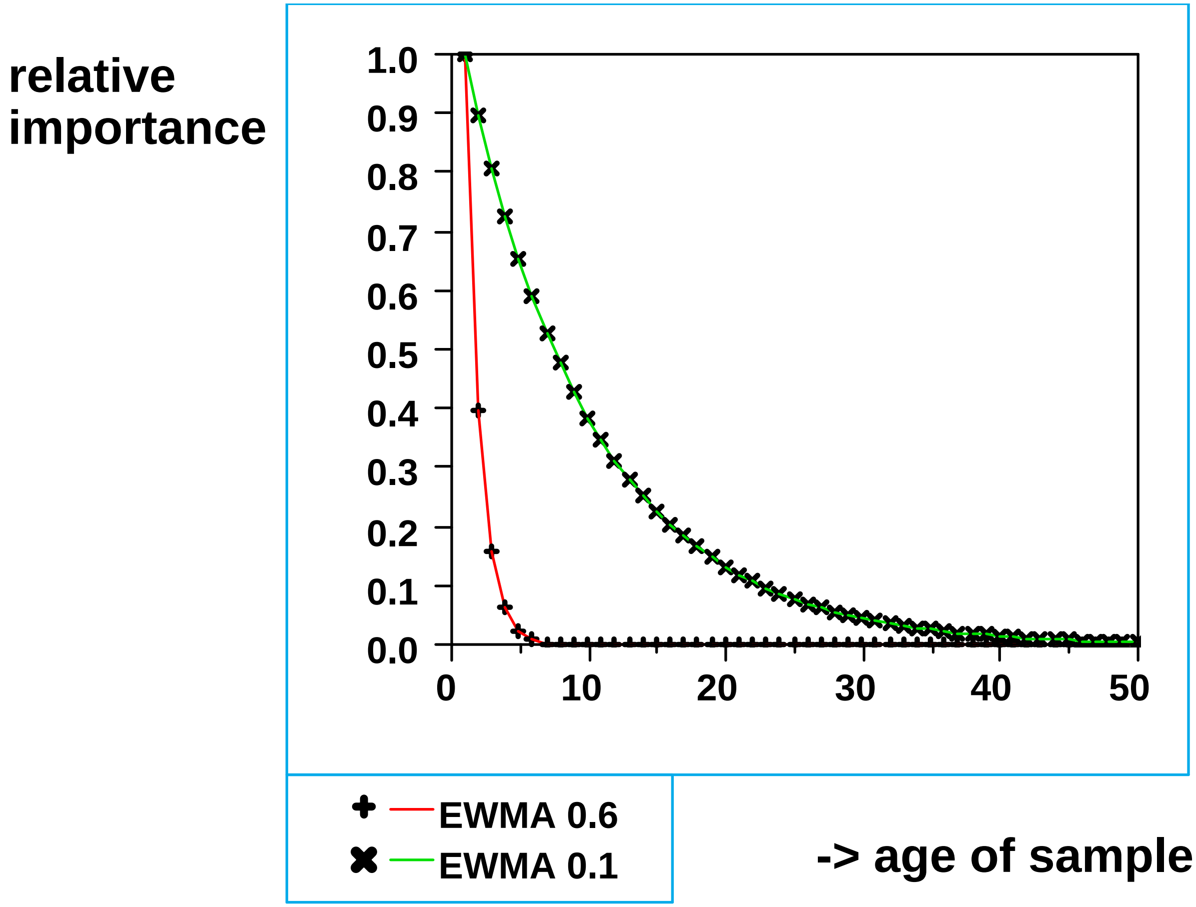
It can be shown that the weights decrease geometrically and that they sum up to unity.

$$z_t = \lambda \sum_{j=0}^{t-1} (1 - \lambda)^j \, \bar{x}_{t-j} + (1 - \lambda)^t z_0$$

$$UCL = \bar{\bar{x}} + 3\sigma \sqrt{\frac{\lambda}{(2 - \lambda) n}}$$

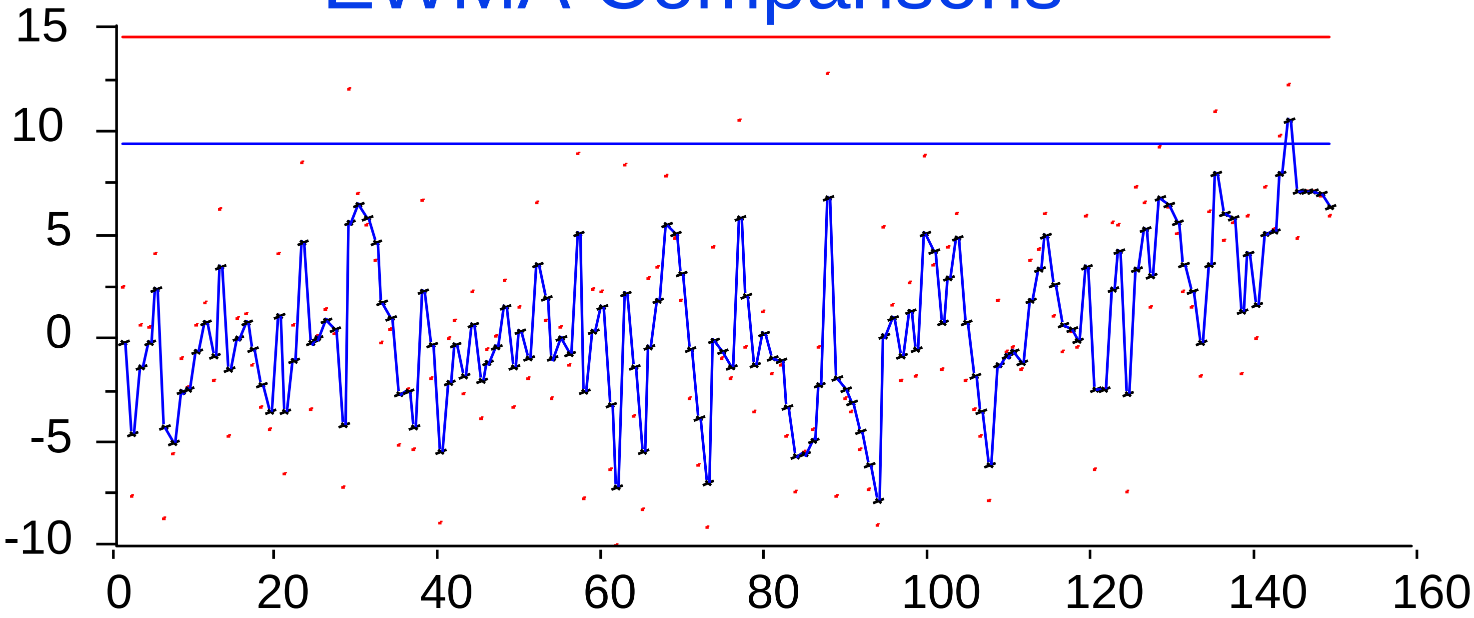
$$LCL = \bar{\bar{x}} - 3\sigma \sqrt{\frac{\lambda}{(2 - \lambda) n}}$$

Two example Weighting Envelopes

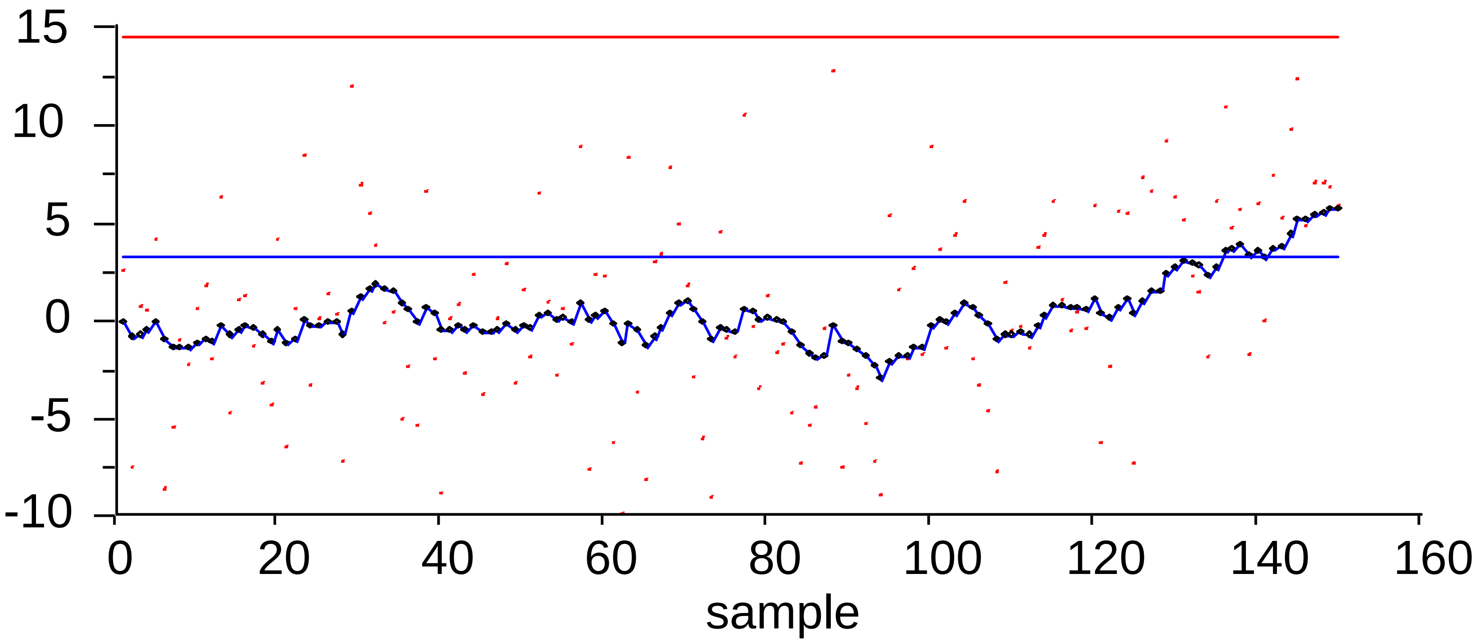


EWMA Comparisons

λ=0.6



λ=0.1



Another View of the EWMA

- The EWMA value z_t is a *forecast* of the sample at the t+1 period.
- Because of this, EWMA belongs to a general category of filters that are known as “time series” filters.

$$\hat{x}_t = f\,(x_{t-1}, x_{t-2}, x_{t-3}, \ldots)$$

$$\hat{x}_t - x_t = a_t$$

Usually:

$$\hat{x}_t = \sum_{i=1}^{p} \phi_i x_{t-i} + \sum_{j=1}^{q} \theta_j a_{t-j}$$

- The proper formulation of these filters can be used for forecasting and feedback / feed-forward control!
- Also, for quality control purposes, these filters can be used to translate a non-IIND signal to an IIND residual...

Summary so far…

While simple control charts are great tools for visualizing the process,it is possible to look at them from another perspective:

Control charts are useful “summaries” of the process statistics.

Charts can be designed to increase sensitivity without sacrificing type I error.

It is this type of advanced charts that can form the foundation of the automation control of the (near) future.

Next stop before we get there: multivariate and model-based SPC!