EECS 290S: Network Information Flow **Fall 2008**

Lecture 27 — December 4

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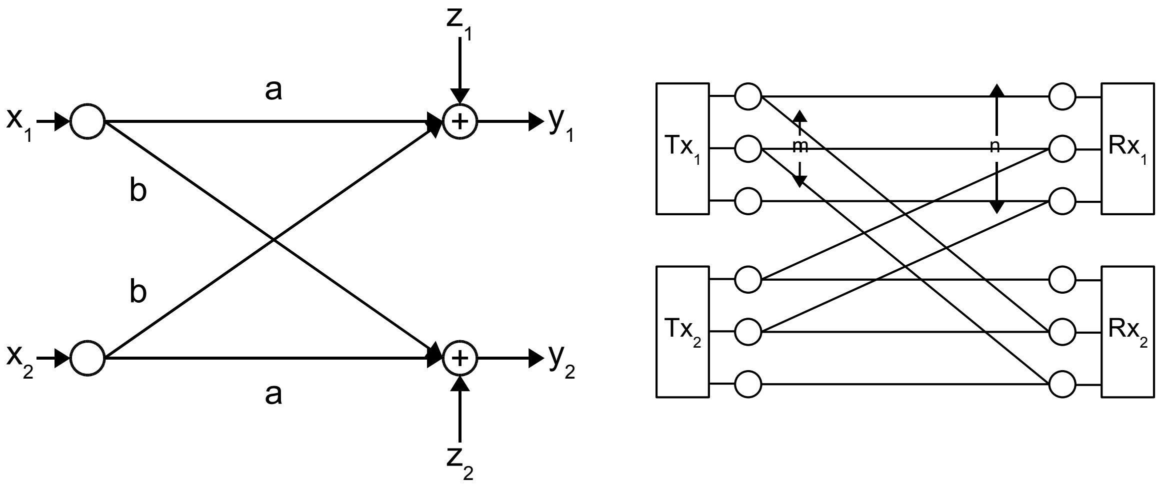
Recap:

For deterministic interference channel, sum capacity

$$C_{sum} = \min\{\max(2m, 2(n-m)), 2n-m\}$$



(a) Symmetric Gaussian interference channel: $|a|^2 = \mathsf{SNR}$, $|b|^2 = \mathsf{INR}$

(b) Corresponding linear deterministic interference channel: $n \leftrightarrow \log_2 \mathsf{SNR}$, $m \leftrightarrow \log_2 \mathsf{INR}$

Figure 27.1. Interference Channels

27.1 Symmetric Gaussian Interference Channel

Without loss of generality, we make $|a|^2 = \mathsf{SNR}$, $|b|^2 = \mathsf{INR}$, set the power constraints to be 1, and normalize the noise variances to unit-variance.

27.2 Achievable Strategy

Split the message into two independent parts: private (m_{ip}) and common (m_{ic}). Encoding is based on two independent Gaussian random codebooks, with power constraints Q_p and

Q_c respectively, for $i = 1, 2$. The power split satisfies $Q_p + Q_c = 1$. For decoding, first both receivers decode (m_{1c}, m_{2c}) simultaneously by treating all private parts as noise, and then decode their own private message m_{ip} after removing all common parts. Choose Q_p such that $|b|^2 Q_p =$ noise variance $= 1$. Therefore, $Q_p = 1/|b|^2$ (assume that $|b|^2 > 1$). The achievable sum rate

$$R_{sum} = \min \left\{ 2\log\left(1 + \mathsf{INR} + \frac{\mathsf{SNR}}{\mathsf{INR}}\right) - 2, \log(1 + \mathsf{SNR} + \mathsf{INR}) + \log\left(2 + \frac{\mathsf{SNR}}{\mathsf{INR}}\right) - 2 \right\}.$$

27.3 Upper Bounds

The side information here is slightly different from that in deterministic interference channel: let

$$\begin{aligned} V_1 &:= bX_1 + Z_2 \\ V_2 &:= bX_2 + Z_1. \end{aligned}$$

Upper bounds can be obtained by giving asymmetric side information to one of the receivers, say, V_1, X_2 to receiver 1. By Fano's inequality and data processing inequality, if a rate pair (R_1, R_2) is achievable,

$$\begin{aligned} N(R_1 + R_2 - \epsilon_N) &\le I(W_1; Y_1^N) + I(W_2; Y_2^N) \le I(X_1^N; Y_1^N) + I(X_2^N; Y_2^N) \\ &\le I(X_1^N; Y_1^N, V_1^N, X_2^N) + I(X_2^N; Y_2^N) \\ &= I(X_1^N; Y_1^N, V_1^N | X_2^N) + I(X_2^N; Y_2^N) \\ &= I(X_1^N; V_1^N | X_2^N) + I(X_1^N; Y_1^N | V_1^N, X_2^N) + I(X_2^N; Y_2^N) \\ &= h(V_1^N) - h(Z_2^N) + h(Y_1^N | V_1^N, X_2^N) - h(Z_1^N) + h(Y_2^N) - h(V_1^N) \\ &= h(Y_2^N) - h(Z_2^N) + h(Y_1^N | V_1^N, X_2^N) - h(Z_1^N) \\ &\le \sum_{i=1}^{N} h(bX_{1i} + aX_{2i} + Z_{2i}) - h(Z_{2i}) + \sum_{i=1}^{N} h(aX_{1i} + Z_{1i} | bX_{1i} + Z_{2i}) - h(Z_{1i}) \\ &\le N\left[\log(1 + \mathsf{SNR} + \mathsf{INR}) + \log\left(1 + \frac{\mathsf{SNR}}{1 + \mathsf{INR}}\right)\right] \end{aligned}$$

Hence, if a rate pair (R_1, R_2) is achievable, then $R_1+R_2 \le \log(1+\mathsf{SNR}+\mathsf{INR})+\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)$. The gap with the achievable sum rate $\log(1+\mathsf{SNR}+\mathsf{INR})+\log\left(2+\frac{\mathsf{SNR}}{\mathsf{INR}}\right)-2$ is

$$\begin{aligned}
&\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)-\log\left(2+\frac{\mathsf{SNR}}{\mathsf{INR}}\right)+2\\
&=\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)-\log\left(1+\frac{\mathsf{SNR}}{2\mathsf{INR}}\right)+1\\
&\le\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)-\log\left(1+\frac{\mathsf{SNR}}{2+2\mathsf{INR}}\right)+1\\
&\le\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)-\log\left(\frac{1}{2}+\frac{1}{2}\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)+1=2.
\end{aligned}$$

Therefore, the gap is at most 2.

On the other hand, upper bounds can also be obtained by feeding symmetric side information to both receivers, say, V_i to receiver i, for $i=1,2$. By Fano's inequality and data processing inequality, if a rate pair (R_1, R_2) is achievable,

$$\begin{aligned}
N(R_1+R_2-\epsilon_N) &\le I(W_1;Y_1^N)+I(W_2;Y_2^N)\le I(X_1^N;Y_1^N)+I(X_2^N;Y_2^N)\\
&\le I(X_1^N;Y_1^N,V_1^N)+I(X_2^N;Y_2^N,V_2^N)\\
&= I(X_1^N;V_1^N)+I(X_2^N;V_2^N)+I(X_1^N;Y_1^N|V_1^N)+I(X_2^N;Y_2^N|V_2^N)\\
&= h(V_1^N)-h(Z_2^N)+h(Y_1^N|V_1^N)-h(V_2^N)+h(V_2^N)-h(Z_1^N)+h(Y_2^N|V_2^N)-h(V_1^N)\\
&= h(Y_1^N|V_1^N)-h(Z_1^N)+h(Y_2^N|V_2^N)-h(Z_2^N)\\
&\le \sum_{i=1}^{N} h(aX_{1i}+bX_{2i}+Z_{1i}|bX_{1i}+Z_{2i})-h(Z_{1i})+\sum_{i=1}^{N} h(bX_{1i}+aX_{2i}+Z_{2i}|bX_{2i}+Z_{1i})-\\
&\le N\left[2\log\left(1+\mathsf{INR}+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)\right]
\end{aligned}$$

Hence, if a rate pair (R_1, R_2) is achievable, then $R_1+R_2\le 2\log\left(1+\mathsf{INR}+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)$. The gap with the achievable sum rate $2\log\left(1+\mathsf{INR}+\frac{\mathsf{SNR}}{\mathsf{INR}}\right)-2$ is

$$\begin{aligned}
&2\log\left(1+\mathsf{INR}+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)-2\log\left(1+\mathsf{INR}+\frac{\mathsf{SNR}}{\mathsf{INR}}\right)+2\\
&\le 2.
\end{aligned}$$

Therefore, the gap is at most 2.

Combining the above two upper bounds, we have a tight characterization of sum capacity, in the sense that the gap between inner and outer bounds is at most 2 bits, for arbitrary $\mathsf{SNR}, \mathsf{INR}$.

Define

$$c(\alpha) := \lim_{\substack{\mathsf{SNR},\mathsf{INR}\to\infty\\ \text{fix } \alpha=\frac{\log_2\mathsf{INR}}{\log_2\mathsf{SNR}}}} \frac{C_{sym}(\mathsf{SNR},\mathsf{INR})}{\log_2\mathsf{SNR}}$$

and its plot s given in Figure **??**, with $\alpha = \frac{\log_2 \mathsf{INR}}{\log_2 \mathsf{SNR}}$ instead of m/n.

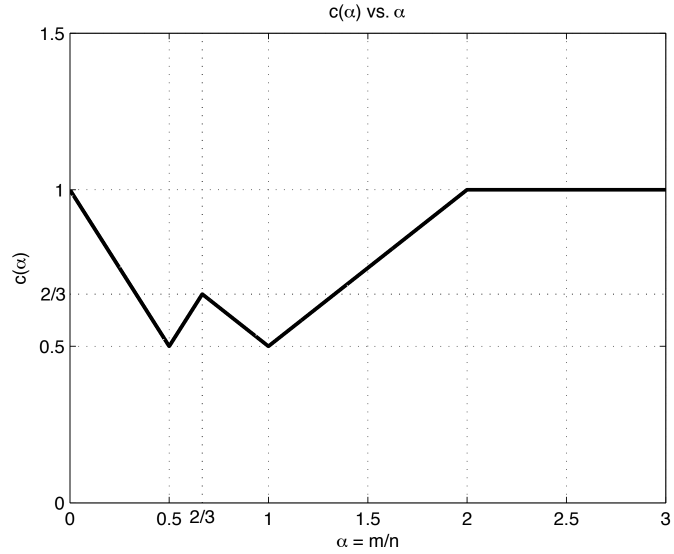


Figure 27.2. $c(\alpha)$

Relook at the second bound, the the genie-aided channel, receiver 1 gets $(aX_1 + bX_2 + Z_1, bX_1 + Z_2)$ and receiver 2 gets $(bX_1 + aX_2 + Z_2, bX_2 + Z_1)$. Our calculation assume that Z_1 and Z_2 are independent. However, the capacity of the original channel does not depend on the joint distribution of (Z_1, Z_2), but the genie-aided one does. Hence, optimizing the correlation between (Z_1, Z_2) can close the gap for $\alpha \in [0, 1/3]$.