EECS 290S: Network Information Flow **Fall 2008**

Lecture 7 — September 23

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7.1 Motivation

Previously covered:

- Max Flow = Min Cut for single source single destination acyclic network.
- Max Flow= Min Cut for multiple destination networks having common message for all destinations.
- Achieving Max Flow using Network Coding.
- Extension to cyclic networks.
- Algorithms for designing network codes.

We now attempt to extend these ideas to more complex scenarios.

1. We can look at more complex traffic patterns, for example multiple destinations with different messages to each destination, and add multiple sources to the mix.
2. So far we have been restricted to error free wireline models, i.e. a node can identify and distinguish the messages it receives on different incoming edges (orthogonal). We can try and look at channel models where this condition does not hold.

If we focus on 2, we find that the constructive proof of the Ford-Fulkerson theorem does not generalize easily to non-orthogonal channels. We will today look at an alternate proof for the Ford-Fulkerson Theorem which lends itself to easier generalization.

7.2 Ford Fulkerson Theorem: Alternate Proof (Alhswede,Cai,Li,Yeung)

Let us look at a general single source - single destination directed acyclic graph with nodes $\in \mathcal{N}$ and edges $\in \mathcal{E}$. There is a single source $s \in \mathcal{N}$ and a single destination $t \in \mathcal{N}$. A directed edge $\in \mathcal{E}$ from i to $j (i, j \in \mathcal{N})$ has a finite capacity c_{ij} symbols/sec, where each symbol is taken from a finite field $\mathbb{F}$. Let $\mathcal{S}$ denote a cut on this graph and $C(\mathcal{S}, \mathcal{S}^c)$ the capacity across the cut as defined in previous lectures.

To show: The capacity C of this network, i.e. the maximum flow from s to t

$$C = \min_{\mathcal{S}} C(\mathcal{S}, \mathcal{S}^c) \tag{7.1}$$

Exercise: A proof for this is given in Yeung's book 1st Edition, section 11.5.1 on pp. 246-250. The proof has an error. Find the error.

A symbol vector originating at s can be written as,

$$\mathbf{x}_s = [x_1, x_2, ..., x_R]^T \tag{7.2}$$

the information rate is thus R. Let us design a coding scheme i.e. a function for every edge $\in \mathcal{E}$. For edges originating at the source s,

$$f_{sj}(\mathbf{x_s}) \in \mathbb{F}^{c_{sj}} \tag{7.3}$$

the function returns c_{sj} symbols on edge $(s, j) \in \mathcal{E}$. At node $i \neq s$ let $\mathbf{x_i}$ be the vector of all incoming symbols into node i,

$$f_{ij}(\mathbf{x_i}) \in \mathbb{F}^{c_{ij}} \tag{7.4}$$

We call these the *local encoding functions.* We can determine the *global encoding function* for a node i such that,

$$F_i(\mathbf{x}_s) = \text{vector of all inputs into node } i$$

i.e. F_i denotes the inputs at node i as a function of original source symbol vector. Since the network is acyclic, we can write the vector observed at destination t as $F_t(\mathbf{x}_s)$ and we wish to decode the source vector $\mathbf{x}_s$ from this. This can be achieved by designing the local encoding functions such that $F_t(\mathbf{x}_s)$ is invertible. Let us chose the local encoding functions to be random.

$$f_{ij} : \forall \mathbf{x}\ f_{ij}(\mathbf{x}) \text{ is uniform in } \mathbb{F}^{c_{ij}}$$

$$P[f_{ij}(\mathbf{x}) = \mathbf{u}] = \frac{1}{|\mathbb{F}|^{c_{ij}}} \forall \mathbf{u} \in \mathbb{F}^{c_{ij}} \tag{7.5}$$

independently chosen across inputs across edges. (Once chosen - the encodings are fixed and are known to all nodes).

Fix $\mathbf{x}_s \neq \mathbf{x}'_s$, node i confuses between $\mathbf{x}_s$ and $\mathbf{x}'_s$ if

$$F_i(\mathbf{x}_s) = F_i(\mathbf{x}'_s)$$

Now P_e the probability of error, i.e. error in choosing random encodings such that the destination confuses between some symbols, can be written using the union bound,

$$P[\text{error}|\mathbf{x}_s] \leq \sum_{\mathbf{x}_s \neq \mathbf{x}'_s} P[F_t(\mathbf{x}_s) = F_t(\mathbf{x}'_s)] \tag{7.6}$$

we can divide the graph into nodes that are confused and nodes that are not confused, this partition can be verified to be a cut

$$\tag{7.7}$$

$$P[F_t(\mathbf{x}_s) = F_t(\mathbf{x}'_s)] = \sum_{\mathcal{S}} P[\text{all nodes in } \mathcal{S} \text{ are not confused and nodes in } \mathcal{S}^c \text{ are confused}] \tag{7.8}$$

For a given partition $\mathcal{S}$ we can topologically sort all nodes $\in \mathcal{S}$ (no cycles) and if we label the source as 1 we can write

$$\mathcal{S} = \{1, 2, 3, ..., |\mathcal{S}|\} \tag{7.9}$$

$\forall i \in \mathcal{S}$ we can define,

$$\tag{7.10}$$

$$A_i = \{F_i(\mathbf{x}_s) \neq F_i(\mathbf{x}'_s)\} \tag{7.11}$$

i.e. A_i is the event that node i is not confused, and,

$$\begin{aligned} B_i = \{f_{ij}(F_i(\mathbf{x}_s)) &= f_{ij}(F_i(\mathbf{x}'_s)) && (7.12)\\ &\forall j \text{ s.t. } (i,j) \text{ is an edge crossing the cut}\} && (7.13) \end{aligned}$$

i.e. B_i is the event that node i is sending confusing information along all edges that cross the cut $\mathcal{S}$. Now we can write,

$$P[\mathcal{S} \text{ not confused and } \mathcal{S}^c \text{ confused}] = P[A_1 A_2 ... A_{|\mathcal{S}|} B_1 B_2 ... B_{|\mathcal{S}|}] \tag{7.14}$$

$$\tag{7.15}$$

i.e. all p in $\mathcal{S}$ are not confused and all information flowing from $\mathcal{S}$ to $\mathcal{S}^c$ causes confusion between $\mathbf{x}_s$ and $\mathbf{x}'_s$.

$$\begin{aligned} P[A_1 A_2 ... A_{|\mathcal{S}|} B_1 B_2 ... B_{|\mathcal{S}|}] &= P[A_1 B_1] P[A_2 B_2 | A_1 B_1] ... && (7.16)\\ &\leq P[B_1|A_1] P[B_2|A_1 A_2 B_1] P[B_3 | A_1 A_2 A_3 B_1 B_2] ... && (7.17) \end{aligned}$$

this follows because, $P[A_1 B_1] = P[B_1|A_1]P[A_1]$ but we know the source node is not confused, $P[A_1] = 1$. Similarly, $P[A_2 B_2 | A_1 B_1] = P[B_1 | A_1 A_2 B_1] P[A_2 | A_1 B_1]$. Now if we define E_S as the set of edges that cross the cut,

$$P[B_1|A_1] = \prod_{j:(1,j)\in E_S} \frac{1}{|\mathbb{F}|^{c_{1j}}} \tag{7.18}$$

and because of independence

$$= \frac{1}{|\mathbb{F}|^{\sum_{j:(1,j)\in E_S} c_{sj}}} \tag{7.19}$$

$$P[B_2|A_1B_1] = P[B_2|A_2] \tag{7.20}$$

$$= \frac{1}{|\mathbb{F}|^{\sum_{j:(2,j)\in E_S} c_{2j}}} \tag{7.21}$$

if we know that node 2 is not confused, the event that it encodes confusing messages on its outgoing edges is independent of what happens on nodes before it in topological order. This gives us a Markov property.

$$P[B_3|A_1A_2A_3B_1B_2] = P[B_3|A_3] \tag{7.22}$$

can be written similarly.

$$P[\mathcal{S} \text{ not confused and } \mathcal{S}^c \text{ confused}] \leq \frac{1}{|\mathbb{F}|^{C(\mathcal{S},\mathcal{S}^c)}} \tag{7.23}$$

$$P[\text{error}|\mathbf{x}_s] \leq \sum_{\mathbf{x}_s \neq \mathbf{x}'_s} P[t \text{ is confused}] \tag{7.24}$$

$$\leq \sum_{\mathbf{x}_s \neq \mathbf{x}'_s} \sum_{\mathcal{S}} |\mathbb{F}|^{-C(\mathcal{S},\mathcal{S}^c)} \tag{7.25}$$

for rate R

$$\leq |\mathbb{F}|^R \sum_{\mathcal{S}} |\mathbb{F}|^{-C(\mathcal{S},\mathcal{S}^c)} \tag{7.26}$$

as long as $R < C(\mathcal{S}, \mathcal{S}^c)$ for every cut (i.e. for the min cut) then

$$P[\text{error}|\mathbf{x}_s] \to 0 \tag{7.27}$$

$$\text{as } |\mathbb{F}| \to \infty \tag{7.28}$$

if $R < \min_{\mathcal{S}} C(\mathcal{S}, \mathcal{S}^c)$

$$\tag{7.29}$$

□

For the multicast problem, we can bound the probability of at least one of the destinations to be in error to be the sum of the probability that each of the destinations to be in error. Each of these terms can be bounded as above to get the desired result.