Epipolar Geometry
class 11

Multiple View Geometry

Comp 290-089

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Multiple View Geometry course schedule

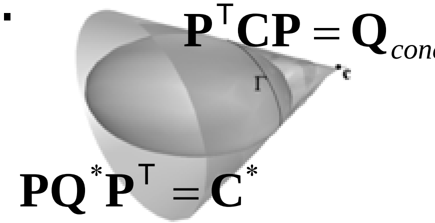
(subject to change)

Jan. 7, 9	Intro & motivation	Projective 2D Geometry
Jan. 14, 16	(no class)	Projective 2D Geometry
Jan. 21, 23	Projective 3D Geometry	(no class)
Jan. 28, 30	Parameter Estimation	Parameter Estimation
Feb. 4, 6	Algorithm Evaluation	Camera Models
Feb. 11, 13	Camera Calibration	Single View Geometry
Feb. 18, 20	<u>Epipolar Geometry</u>	3D reconstruction
Feb. 25, 27	Fund. Matrix Comp.	Structure Comp.
Mar. 4, 6	Planes & Homographies	Trifocal Tensor
Mar. 18, 20	Three View Reconstruction	Multiple View Geometry
Mar. 25, 27	MultipleView Reconstruction	Bundle adjustment
Apr. 1, 3	Auto-Calibration	Papers
Apr. 8, 10	Dynamic SfM	Papers
Apr. 15, 17	Cheirality	Papers
Apr. 22, 24	Duality	Project Demos

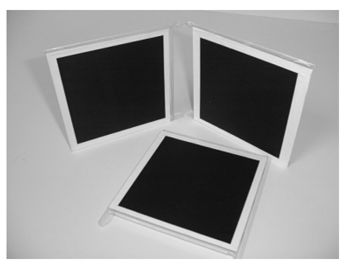
More Single-View Geometry

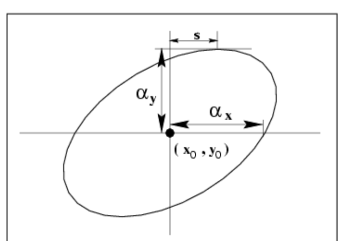
- Projective cameras and planes, lines, conics and quadrics.

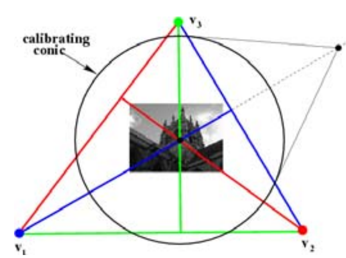
$$\Pi = \mathrm{P}^{\mathsf{T}}\mathrm{l}$$



- Camera calibration and vanishing points, calibrating conic and the IAC







Two-view geometry

Epipolar geometry

3D reconstruction

F-matrix comp.

Structure comp.

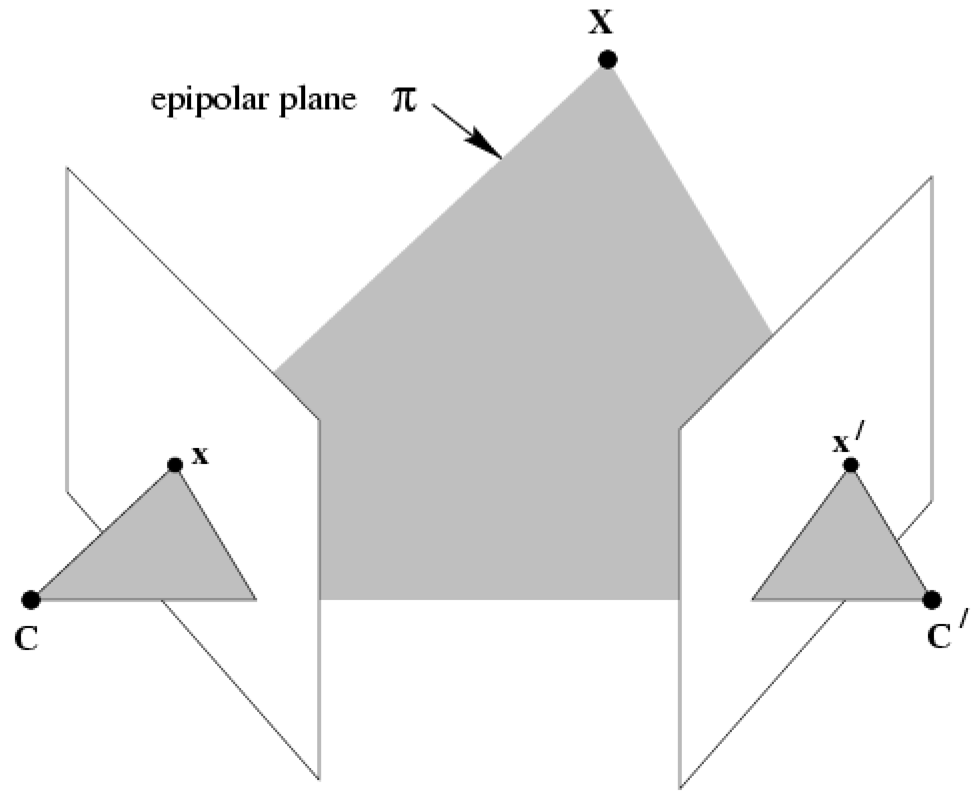
Three questions:

(i) Correspondence geometry: Given an image point x in the first view, how does this constrain the position of the corresponding point x' in the second image?

(ii) Camera geometry (motion): Given a set of corresponding image points $\{x_i \leftrightarrow x'_i\}$, $i=1,\ldots,n$, what are the cameras P and P' for the two views?

(iii) Scene geometry (structure): Given corresponding image points $x_i \leftrightarrow x'_i$ and cameras P, P', what is the position of (their pre-image) X in space?

The epipolar geometry

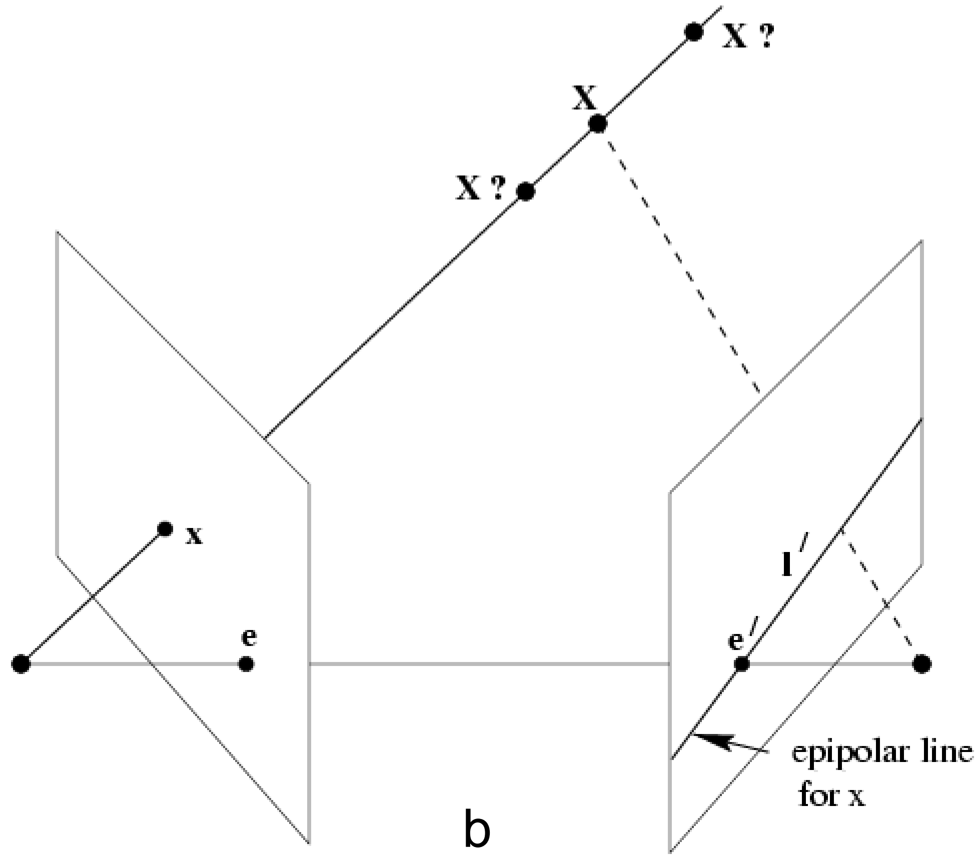


(a)

C,C',x,x' and X are coplanar

Fig. 9.1. **Point correspondence geometry.** *(a) The two cameras are indicated by their centres* **C** *and* **C**$'$ *and image planes. The camera centres, 3-space point* **X**, *and its images* **x** *and* **x**$'$ *lie in a common plane* π. *(b) An image point* **x** *back-projects to a ray in 3-space defined by the first camera centre,* **C**, *and* **x**. *This ray is imaged as a line* l' *in the second view. The 3-space point* **X** *which projects to* **x** *must lie on this ray, so the image of* **X** *in the second view must lie on* l'.

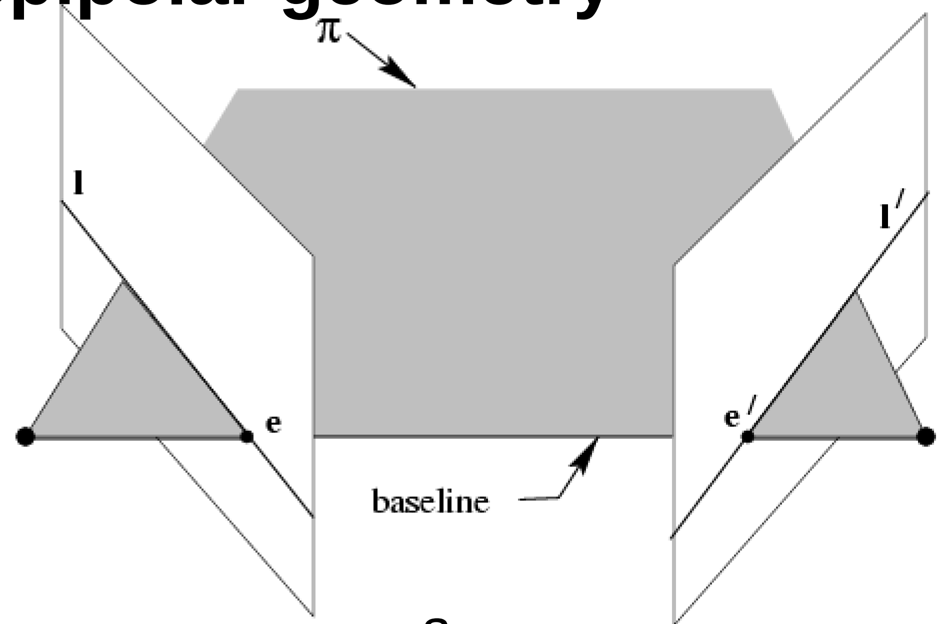
The epipolar geometry



What if only C,C',x are known?

Fig. 9.1. **Point correspondence geometry.** *(a) The two cameras are indicated by their centres* $\mathbf{C}$ *and* $\mathbf{C}'$ *and image planes. The camera centres, 3-space point* $\mathbf{X}$*, and its images* $\mathbf{x}$ *and* $\mathbf{x}'$ *lie in a common plane* $\boldsymbol{\pi}$*. (b) An image point* $\mathbf{x}$ *back-projects to a ray in 3-space defined by the first camera centre,* $\mathbf{C}$*, and* $\mathbf{x}$*. This ray is imaged as a line* $\mathbf{l}'$ *in the second view. The 3-space point* $\mathbf{X}$ *which projects to* $\mathbf{x}$ *must lie on this ray, so the image of* $\mathbf{X}$ *in the second view must lie on* $\mathbf{l}'$*.*

The epipolar geometry

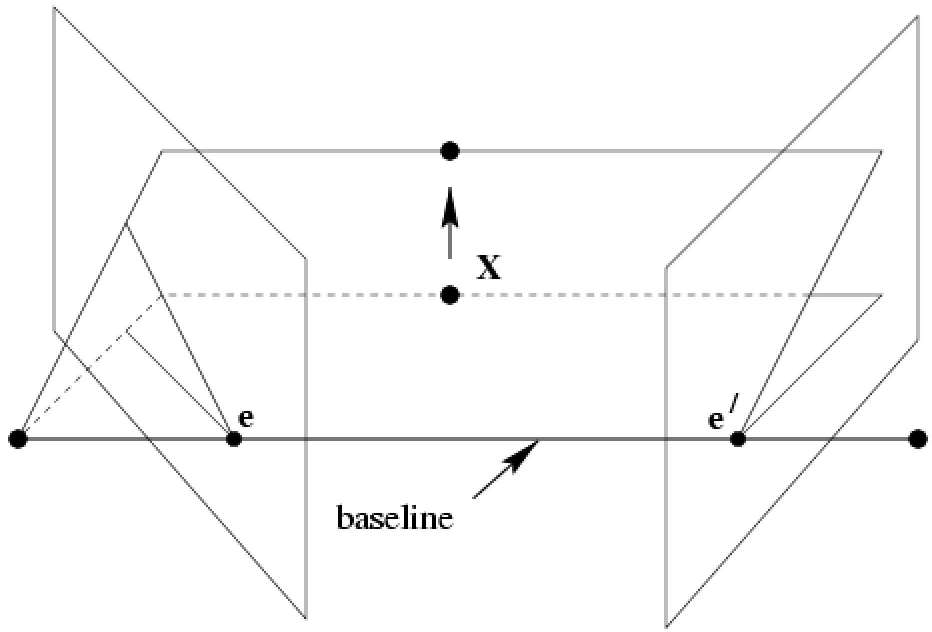


a

All points on π project on l and l'

Fig. 9.2. **Epipolar geometry.** *(a) The camera baseline intersects each image plane at the epipoles* $\mathbf{e}$ *and* $\mathbf{e}'$*. Any plane* π *containing the baseline is an epipolar plane, and intersects the image planes in corresponding epipolar lines* $\mathbf{l}$ *and* $\mathbf{l}'$*. (b) As the position of the 3D point* $\mathbf{X}$ *varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.*

The epipolar geometry



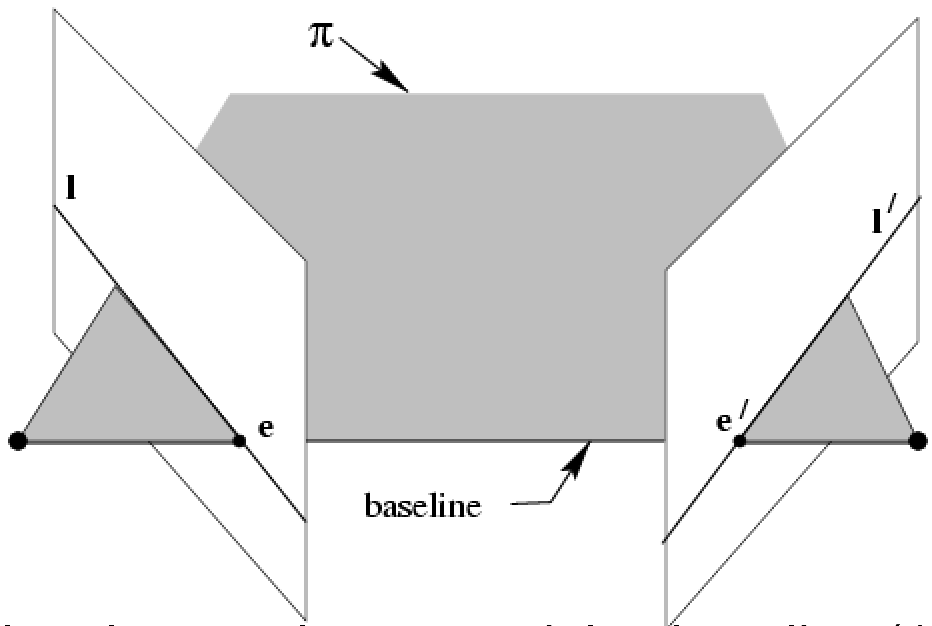
b

Family of planes π and lines l and l'
Intersection in e and e'

Fig. 9.2. **Epipolar geometry.** *(a) The camera baseline intersects each image plane at the epipoles* $\mathbf{e}$ *and* $\mathbf{e}'$. *Any plane* π *containing the baseline is an epipolar plane, and intersects the image planes in corresponding epipolar lines* $\mathbf{l}$ *and* $\mathbf{l}'$. *(b) As the position of the 3D point* $\mathbf{X}$ *varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.*

The epipolar geometry

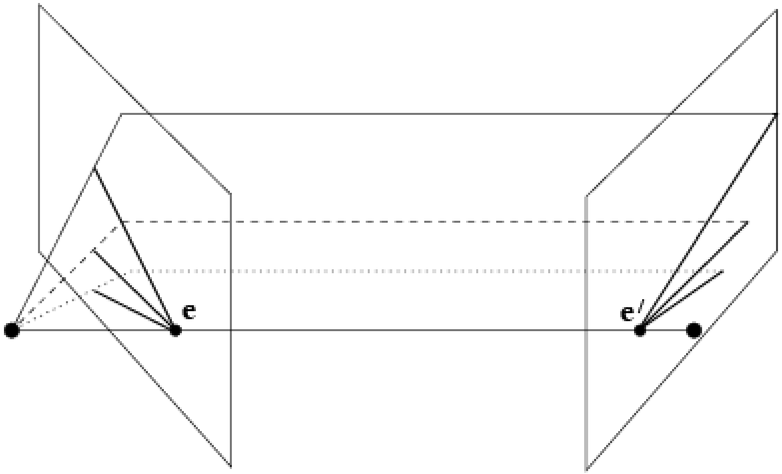
epipoles e,e’
= intersection of baseline with image plane
= projection of projection center in other image
= vanishing point of camera motion direction



an epipolar plane = plane containing baseline (1-D family)

an epipolar line = intersection of epipolar plane with image
(always come in corresponding pairs)

Example: converging cameras



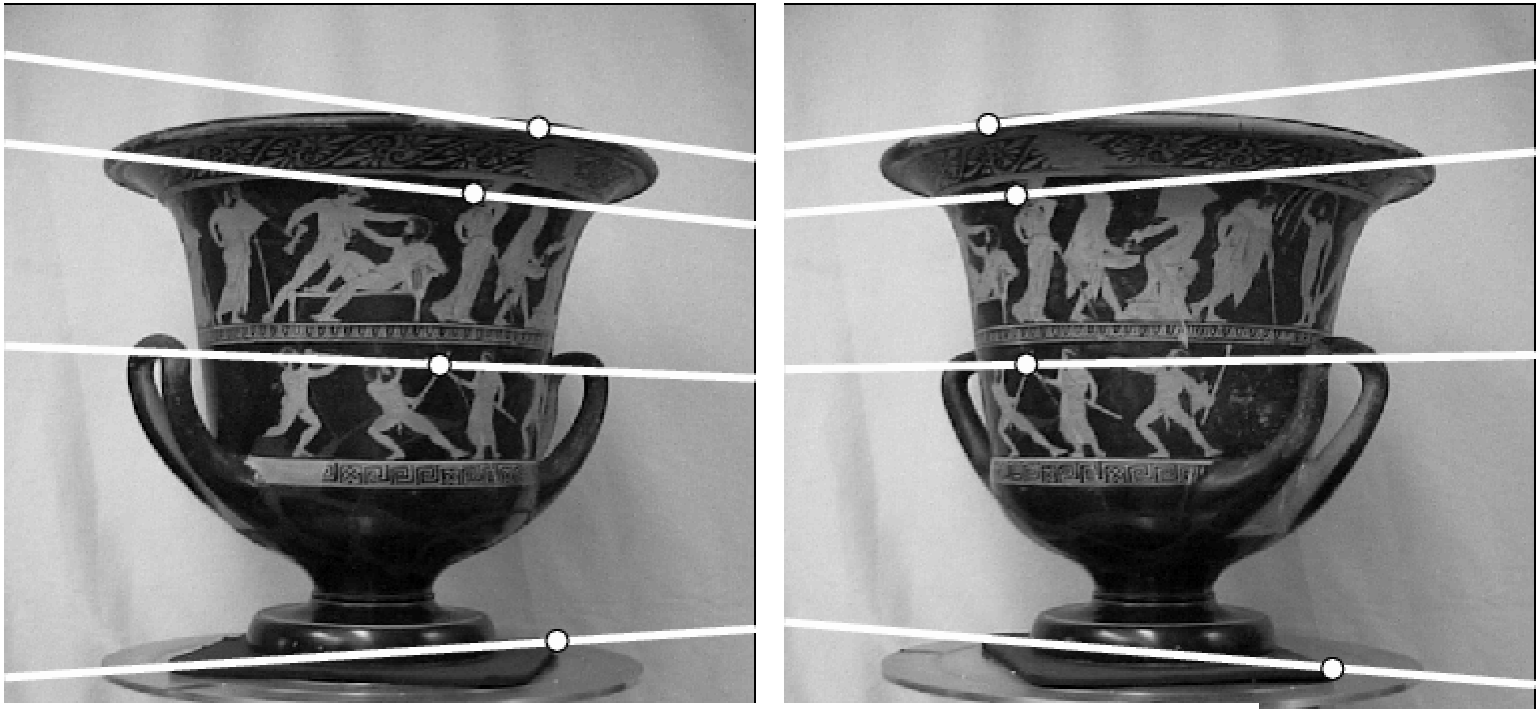


Fig. 9.3. **Converging cameras.** *(a) Epipolar geometry for converging cameras. (b) and (c) A pair of images with superimposed corresponding points and their epipolar lines (in white). The motion between the views is a translation and rotation. In each image, the direction of the other camera may be inferred from the intersection of the pencil of epipolar lines. In this case, both epipoles lie outside of the visible image.*

Example: motion parallel with image plane

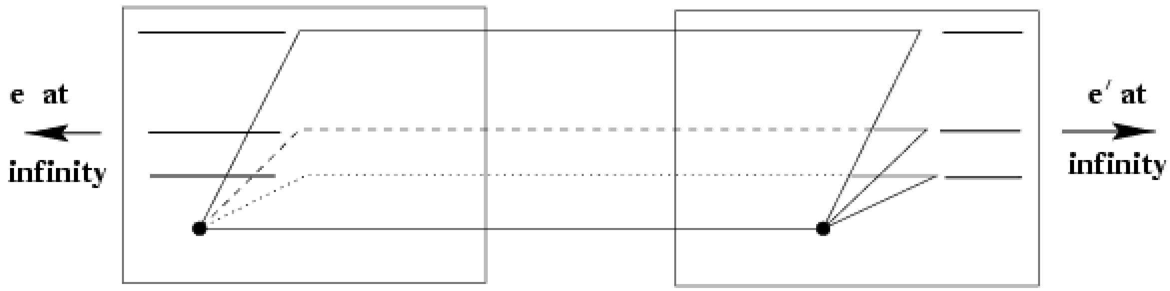


Fig. 9.4. **Motion parallel to the image plane.** *In the case of a special motion where the translation is parallel to the image plane, and the rotation axis is perpendicular to the image plane, the intersection of the baseline with the image plane is at infinity. Consequently the epipoles are at infinity, and epipolar lines are parallel. (a) Epipolar geometry for motion parallel to the image plane. (b) and (c) a pair of images for which the motion between views is (approximately) a translation parallel to the x-axis, with no rotation. Four corresponding epipolar lines are superimposed in white. Note that corresponding points lie on corresponding epipolar lines.*

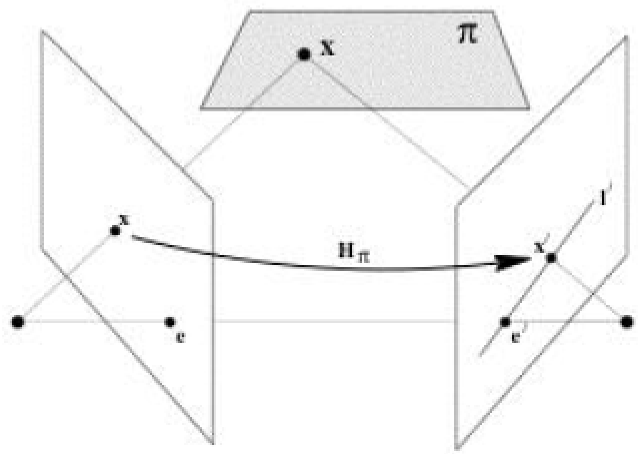


Fig. 9.5. *A point* $\mathbf{x}$ *in one image is transferred via the plane* π *to a matching point* $\mathbf{x}'$ *in the second image. The epipolar line through* $\mathbf{x}'$ *is obtained by joining* $\mathbf{x}'$ *to the epipole* $\mathbf{e}'$. *In symbols one may write* $\mathbf{x}' = \mathtt{H}_\pi \mathbf{x}$ *and* $\mathbf{l}' = [\mathbf{e}']_\times \mathbf{x}' = [\mathbf{e}']_\times \mathtt{H}_\pi \mathbf{x} = \mathtt{F}\mathbf{x}$ *where* $\mathtt{F} = [\mathbf{e}']_\times \mathtt{H}_\pi$ *is the fundamental matrix.*

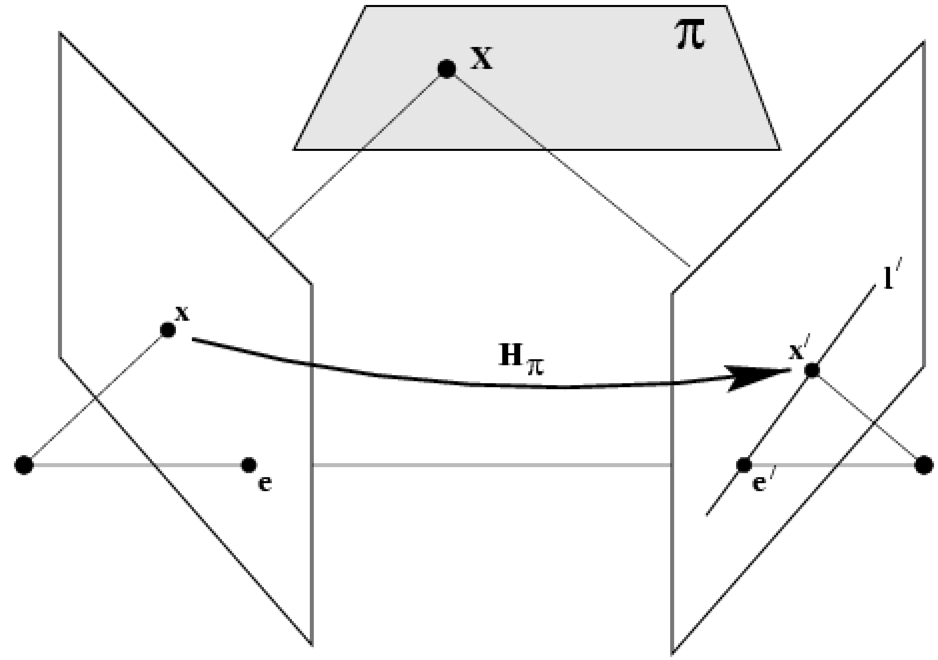
The fundamental matrix F

algebraic representation of epipolar geometry

$$\mathrm{x} \mapsto \mathrm{l}'$$

we will see that mapping is (singular) correlation
(i.e. projective mapping from points to lines)
represented by the fundamental matrix F

The fundamental matrix F
geometric derivation



$$x' = H_\pi x$$

$$l' = e' \times x' = [e']_\times H_\pi x = Fx$$

mapping from 2-D to 1-D family (rank 2)

Result 9.1. *The fundamental matrix* F *may be written as* $F = [e']_\times H_\pi$, *where* H_π *is the transfer mapping from one image to another via any plane* π. *Furthermore, since* $[e']_\times$ *has rank 2 and* H_π *rank 3,* F *is a matrix of rank 2.*

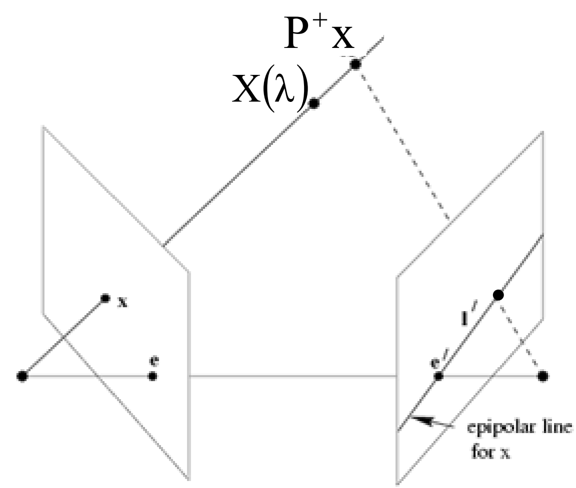
The fundamental matrix F

algebraic derivation

$$X(\lambda) = P^+x + \lambda C \qquad \left(P^+P = I\right)$$

$$l = P'C \times P'P^+x$$

$$F = [e']_\times P'P^+$$



(note: doesn't work for C=C' $\Rightarrow$ F=0)

The fundamental matrix F

correspondence condition

The fundamental matrix satisfies the condition that for any pair of corresponding points $x \leftrightarrow x'$ in the two images

$$x'^T F x = 0 \qquad \left(x'^T l' = 0\right)$$

Result 9.3. *The fundamental matrix satisfies the condition that for any pair of corresponding points* $\mathbf{x} \leftrightarrow \mathbf{x}'$ *in the two images*

$$\mathbf{x}'^{\mathsf{T}} \mathrm{F} \mathbf{x} = 0.$$

The fundamental matrix F

F is the unique 3x3 rank 2 matrix that satisfies $x'^T F x = 0$ for all $x \leftrightarrow x'$

(i) **Transpose:** if F is fundamental matrix for (P,P'), then F^T is fundamental matrix for (P',P)

(ii) **Epipolar lines:** $l' = Fx$ & $l = F^T x'$

(iii) **Epipoles:** on all epipolar lines, thus $e'^T F x = 0, \forall x$ $\Rightarrow e'^T F = 0$, similarly $Fe = 0$

(iv) **F** has 7 d.o.f. , i.e. 3x3-1(homogeneous)-1(rank2)

(v) **F** is a correlation, projective mapping from a point x to a line $l' = Fx$ (not a proper correlation, i.e. not invertible)

Epipolar Line Homography

9.3 Fundamental matrices arising from special motions 247

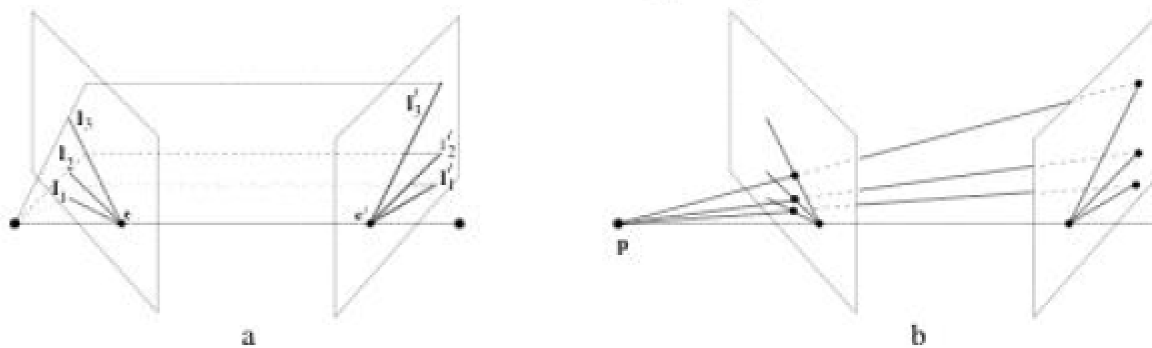
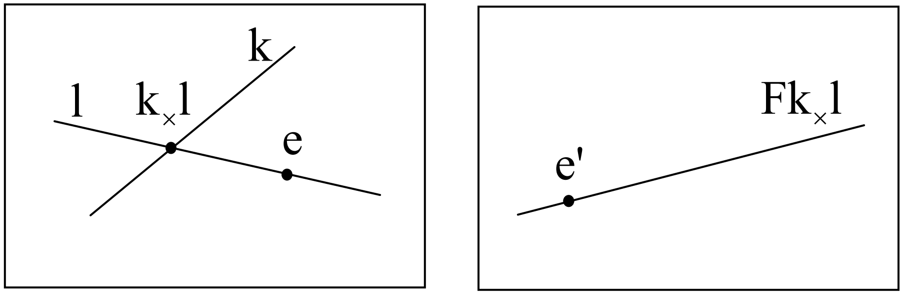


Fig. 9.6. **Epipolar line homography.** *(a) There is a pencil of epipolar lines in each image centred on the epipole. The correspondence between epipolar lines,* $\mathbf{l}_i \leftrightarrow \mathbf{l}'_i$, *is defined by the pencil of planes with axis the baseline. (b) The corresponding lines are related by a perspectivity with centre any point* $\mathbf{p}$ *on the baseline. It follows that the correspondence between epipolar lines in the pencils is a 1D homography.*

The epipolar line geometry

l,l' epipolar lines, k line not through e
$\Rightarrow$ $l'=F[k]_x l$ and symmetrically $l=F^T[k']_x l'$



(pick k=e, since $e^Te\neq0$)

$$l' = F[e]_\times l \qquad l = F^T[e']_\times l'$$

Result 9.5. *Suppose* $\mathbf{l}$ *and* $\mathbf{l}'$ *are corresponding epipolar lines, and* $\mathbf{k}$ *is any line not passing through the epipole* $\mathbf{e}$*, then* $\mathbf{l}$ *and* $\mathbf{l}'$ *are related by* $\mathbf{l}' = \mathtt{F}[\mathbf{k}]_\times \mathbf{l}$*. Symmetrically,* $\mathbf{l} = \mathtt{F}^\mathsf{T}[\mathbf{k}']_\times \mathbf{l}'$.

Pure Translation camera motion

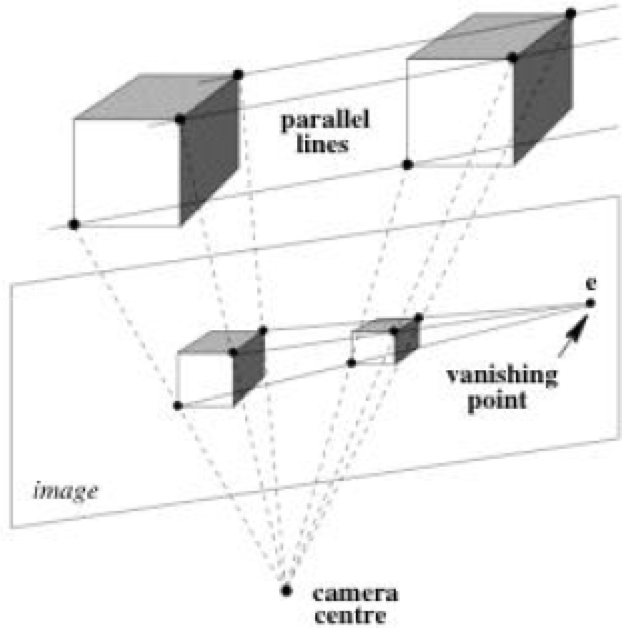
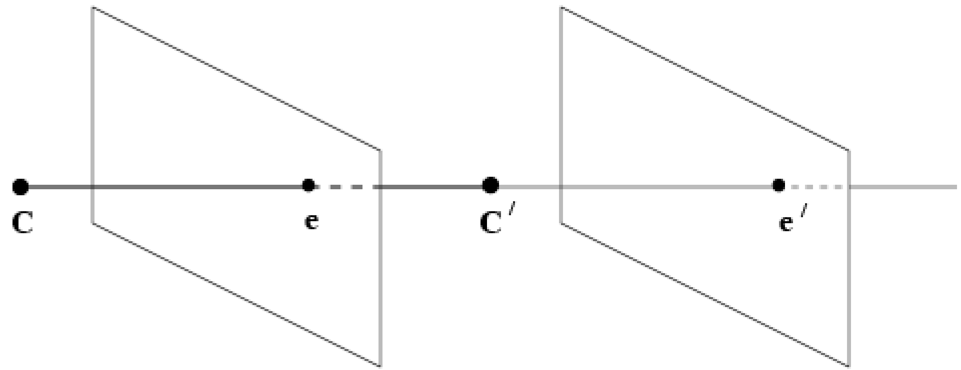


Fig. 9.7. *Under a pure translational camera motion, 3D points appear to slide along parallel rails. The images of these parallel lines intersect in a vanishing point corresponding to the translation direction. The epipole* **e** *is the vanishing point.*

Fundamental matrix for pure translation



Forward motion

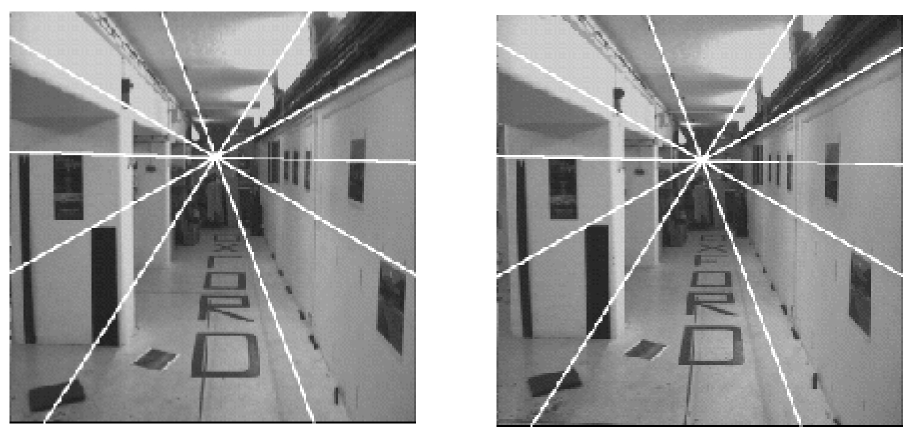


Fig. 9.8. **Pure translational motion.** *(a) under the motion the epipole is a fixed point, i.e. has the same coordinates in both images, and points appear to move along lines radiating from the epipole. The epipole in this case is termed the* Focus of Expansion *(FOE). (b) and (c) the same epipolar lines are overlaid in both cases. Note the motion of the posters on the wall which slide along the epipolar line.*

Fundamental matrix for pure translation

$$F = [e']_\times H_\infty = [e']_\times \qquad (H_\infty = K^{-1}RK)$$

example:

$$e' = (1,0,0)^\mathsf{T} \qquad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}_\times$$

$$x'^\mathsf{T} F x = 0 \Leftrightarrow y = y'$$

$$x = PX = K[I\,|\,0]X \qquad (X,Y,Z)^\mathsf{T} = K^{-1}x/Z$$

$$x' = P'X = K[I\,|\,t]\begin{bmatrix} K^{-1}x \\ Z \end{bmatrix} \qquad x' = x + Kt/Z$$

motion starts at x and moves towards e, faster depending on *Z*

pure translation: F only 2 d.o.f., $x^\mathsf{T}[e]_x x=0 \Rightarrow$ auto-epipolar

General motion

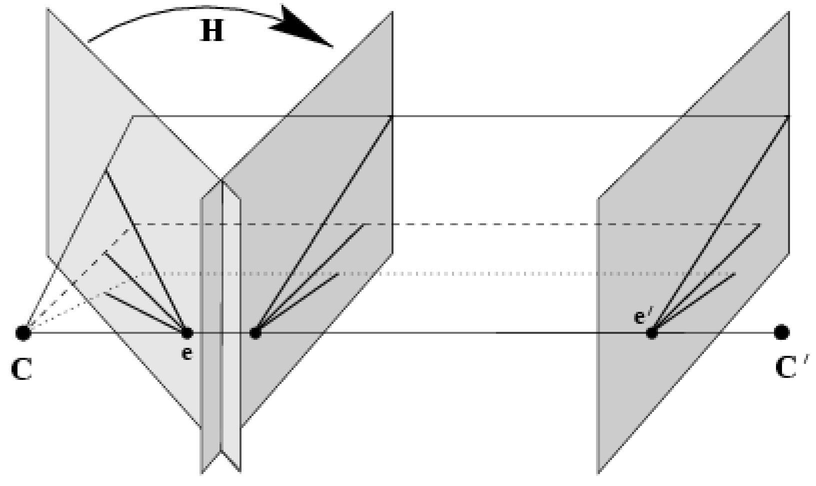


Fig. 9.9. **General camera motion.** *The first camera (on the left) may be rotated and corrected to simulate a pure translational motion. The fundamental matrix for the original pair is the product* $F = [e']_\times H$*, where* $[e']_\times$ *is the fundamental matrix of the translation, and* H *is the projective transformation corresponding to the correction of the first camera.*

$$x'^T [e']_\times H x = 0$$

$$x'^T [e']_\times \hat{x} = 0$$

$$x' = K' R K^{-1} x + K' t/Z$$

Geometric representation of F

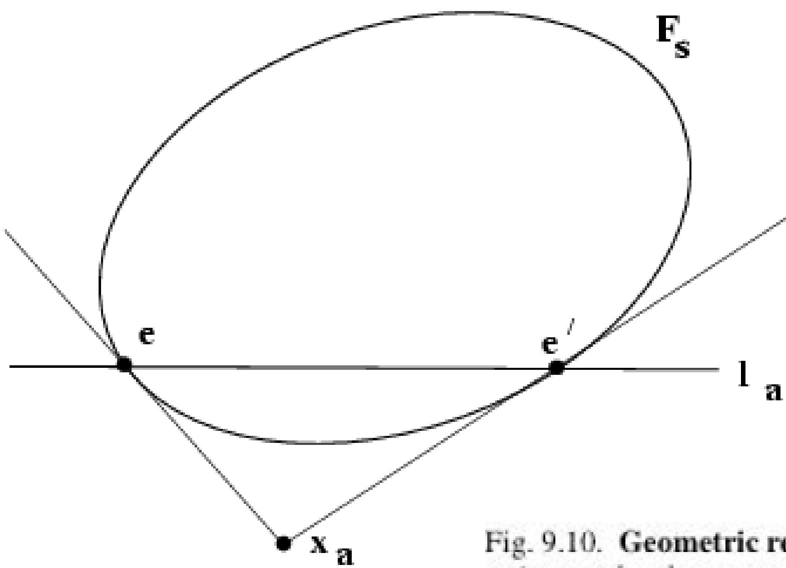
$$F_S = \left(F + F^T\right)/2 \qquad F_A = \left(F - F^T\right)/2 \qquad \left(F = F_S + F_A\right)$$

$$x \leftrightarrow x \qquad x^T F x = 0 \qquad \left(x^T F_A x \equiv 0\right)$$

$$x^T F_S x = 0$$

F_s: Steiner conic, 5 d.o.f.

$F_a=[x_a]_x$: pole of line ee’ w.r.t. F_s, 2 d.o.f.



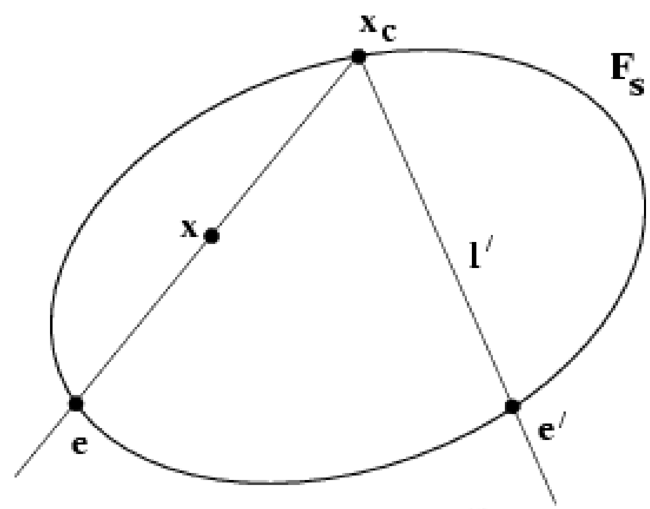
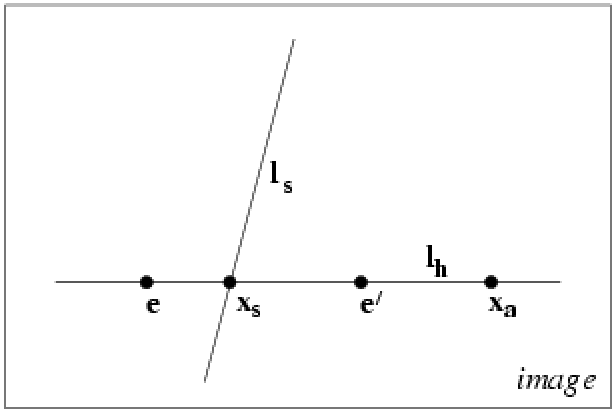


Fig. 9.10. **Geometric representation of F.** *(a) The conic* F_S *represents the symmetric part of* F, *and the point* x_a *the skew-symmetric part. The conic* F_S *is the locus of intersection of corresponding epipolar lines, assuming both images are overlaid on top of each other. It is the image of the horopter curve. The line* l_a *is the polar of* x_a *with respect to the conic* F_S. *It intersects the conic at the epipoles* e *and* e′. *(b) The epipolar line* l′ *corresponding to a point* x *is constructed as follows: intersect the line defined by the points* e *and* x *with the conic. This intersection point is* x_c. *Then* l′ *is the line defined by the points* x_c *and* e′.

Pure planar motion

Steiner conic F_s is degenerate (two lines)



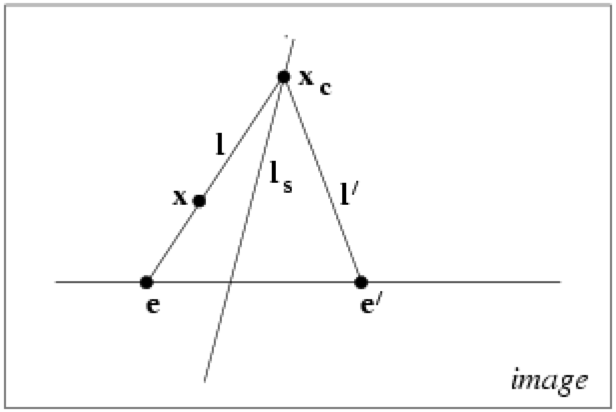


Fig. 9.11. **Geometric representation of F for planar motion.** *(a) The lines* $\mathbf{l}_s$ *and* $\mathbf{l}_h$ *constitute the Steiner conic for this motion, which is degenerate. Compare this figure with the conic for general motion shown in figure 9.10. (b) The epipolar line* $\mathbf{l}'$ *corresponding to a point* $\mathbf{x}$ *is constructed as follows: intersect the line defined by the points* $\mathbf{e}$ *and* $\mathbf{x}$ *with the (conic) line* $\mathbf{l}_s$*. This intersection point is* $\mathbf{x}_c$*. Then* $\mathbf{l}'$ *is the line defined by the points* $\mathbf{x}_c$ *and* $\mathbf{e}'$*.*

Projective transformation and invariance

Derivation based purely on projective concepts

$$\hat{x} = Hx,\ \hat{x}' = H'x' \Rightarrow \hat{F} = H'^{-T} F H^{-1}$$

F invariant to transformations of projective 3-space

Result 9.8. *If* H *is a* 4×4 *matrix representing a projective transformation of 3-space, then the fundamental matrices corresponding to the pairs of camera matrices* (P, P') *and* $(PH, P'H)$ *are the same.*

$$(P, P') \mapsto F \qquad \text{unique}$$

$$F \mapsto (P, P') \qquad \text{not unique}$$

canonical form

$$\begin{aligned} P &= [I \mid 0] \\ P' &= [M \mid m] \end{aligned} \qquad F = [m]_\times M$$

Result 9.9. *The fundamental matrix corresponding to a pair of camera matrices* $P = [I \mid 0]$ *and* $P' = [M \mid \mathbf{m}]$ *is equal to* $[\mathbf{m}]_\times M$.

Projective ambiguity of cameras given F

previous slide: at least projective ambiguity
this slide: not more!

Show that if F is same for (P,P') and ($\tilde{P}$,$\tilde{P}$'),
there exists a projective transformation H so that
$\tilde{P}$=HP and $\tilde{P}$'=HP'

$$\mathrm{P} = [\mathrm{I} \mid 0] \quad \mathrm{P'} = [\mathrm{A} \mid \mathrm{a}] \quad \tilde{\mathrm{P}} = [\mathrm{I} \mid 0] \quad \tilde{\mathrm{P}}' = [\tilde{\mathrm{A}} \mid \tilde{\mathrm{a}}]$$

$$\mathrm{F} = [\mathrm{a}]_\times \mathrm{A} = [\tilde{\mathrm{a}}]_\times \tilde{\mathrm{A}}$$

Theorem 9.10. *Let* F *be a fundamental matrix and let* $(\mathrm{P}, \mathrm{P}')$ *and* $(\tilde{\mathrm{P}}, \tilde{\mathrm{P}}')$ *be two pairs of camera matrices such that* F *is the fundamental matrix corresponding to each of these pairs. Then there exists a non-singular* 4×4 *matrix* H *such that* $\tilde{\mathrm{P}} = \mathrm{PH}$ *and* $\tilde{\mathrm{P}}' = \mathrm{P'H}$.

<u>lemma:</u> $\tilde{\mathrm{a}} = k\mathrm{a} \quad \tilde{\mathrm{A}} = k^{-1}\left(\mathrm{A} + \mathrm{av}^{\mathrm{T}}\right)$

Lemma 9.11. *Suppose the rank 2 matrix* F *can be decomposed in two different ways as* $\mathrm{F} = [\mathbf{a}]_\times \mathrm{A}$ *and* $\mathrm{F} = [\tilde{\mathbf{a}}]_\times \tilde{\mathrm{A}}$*; then* $\tilde{\mathbf{a}} = k\mathbf{a}$ *and* $\tilde{\mathrm{A}} = k^{-1}(\mathrm{A} + \mathbf{a}\mathbf{v}^{\mathsf{T}})$ *for some non-zero constant* k *and 3-vector* $\mathbf{v}$.

Canonical cameras given F

Result 9.12. *A non-zero matrix F is the fundamental matrix corresponding to a pair of camera matrices P and P' if and only if $P'^{\mathsf{T}}FP$ is skew-symmetric.*

F matrix corresponds to P,P' iff $P'^{T}FP$ is skew-symmetric

$$\left(X^{T}P'^{T}\,FPX = 0, \forall X\right)$$

Result 9.13. *Let F be a fundamental matrix and S any skew-symmetric matrix. Define the pair of camera matrices*

$$P = [I \mid 0] \quad \textit{and} \quad P' = [SF \mid e'],$$

where e' is the epipole such that $e'^{\mathsf{T}}F = 0$, and assume that P' so defined is a valid camera matrix (has rank 3). Then F is the fundamental matrix corresponding to the pair (P, P').

Possible choice:

$$P = [I \mid 0] \quad P' = [[e']_{\times}F \mid e']$$

Result 9.14. *The camera matrices corresponding to a fundamental matrix F may be chosen as $P = [I \mid 0]$ and $P' = [[e']_{\times}F \mid e']$.*

Result 9.15. *The general formula for a pair of canonic camera matrices corresponding to a fundamental matrix F is given by*

$$P = [I \mid 0] \quad P' = [[e']_{\times}F + e'v^{\mathsf{T}} \mid \lambda e'] \tag{9.10}$$

where v is any 3-vector, and λ a non-zero scalar.

The essential matrix

~fundamental matrix for calibrated cameras (remove K)

$$E = [t]_\times R = R[R^T t]_\times$$

$$\hat{x}'^T E \hat{x} = 0 \qquad \left(\hat{x} = K^{-1}x;\ \hat{x}' = K^{-1}x'\right)$$

$$E = K'^T F K$$

5 d.o.f. (3 for R; 2 for t up to scale)

Result 9.17. *A 3×3 matrix is an essential matrix if and only if two of its singular values are equal, and the third is zero.*

E is essential matrix if and only if
two singularvalues are equal (and third=0)

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$E = U\mathrm{diag}(1,1,0)V^T$$

Result 9.18. *Suppose that the SVD of* E *is* $U\,\mathrm{diag}(1,1,0)V^T$. *Using the notation of (9.13), there are (ignoring signs) two possible factorizations* $E = SR$ *as follows:*

$$S = UZU^T \quad R = UWV^T \ \ or\ \ UW^TV^T\ . \tag{9.14}$$

Four possible reconstructions from E

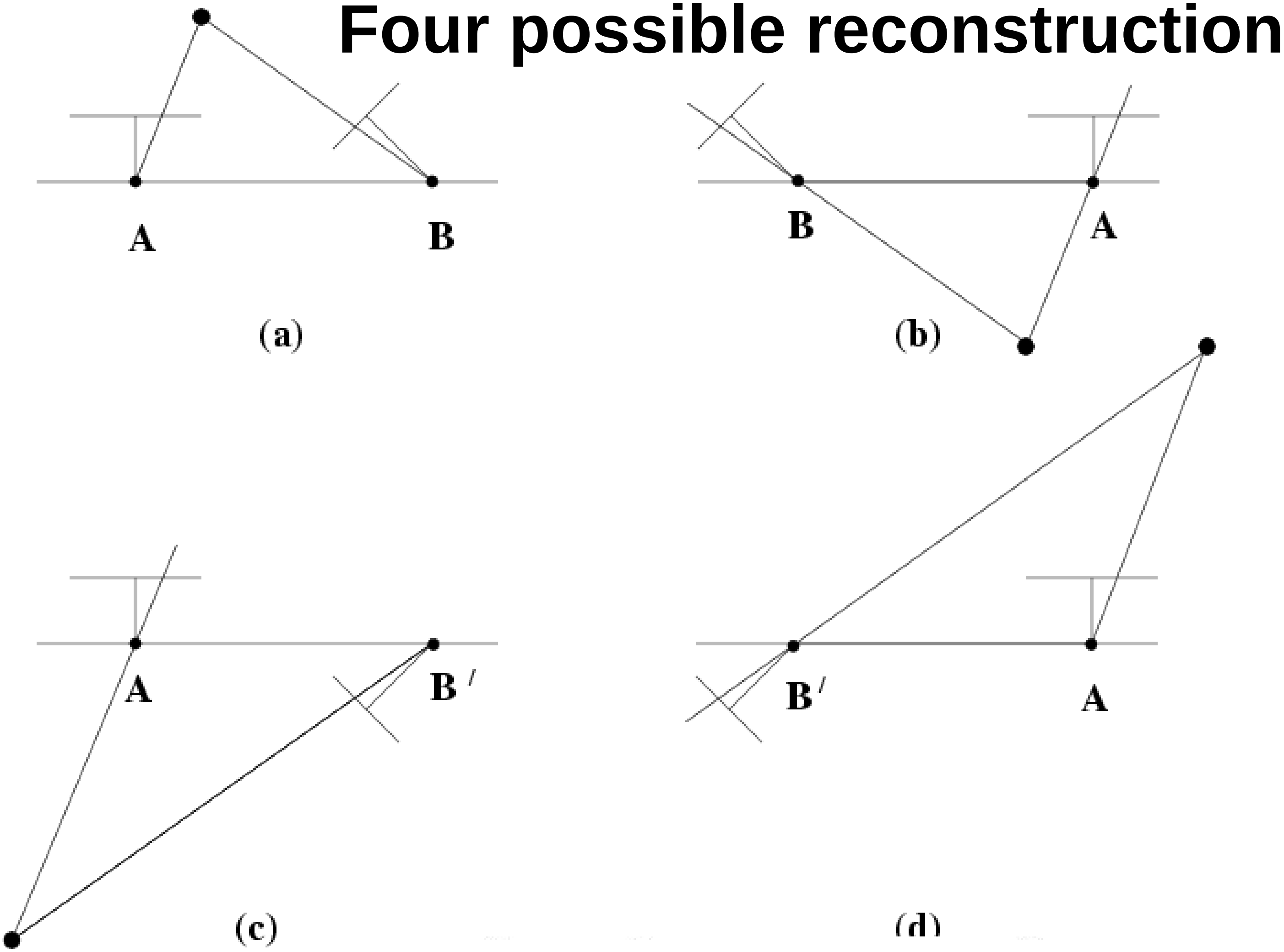


Fig. 9.12. **The four possible solutions for calibrated reconstruction from E.** *Between the left and right sides there is a baseline reversal. Between the top and bottom rows camera B rotates 180° about the baseline. Note, only in (a) is the reconstructed point in front of both cameras.*

Result 9.19. *For a given essential matrix* $E = U\,\mathrm{diag}(1,1,0)V^T$, *and first camera matrix* $P = [I \mid 0]$, *there are four possible choices for the second camera matrix* P', *namely*

$$P' = [UWV^T \mid +\mathbf{u}_3] \text{ or } [UWV^T \mid -\mathbf{u}_3] \text{ or } [UW^TV^T \mid +\mathbf{u}_3] \text{ or } [UW^TV^T \mid -\mathbf{u}_3].$$

(only one solution where points is in front of both cameras)

Next class: 3D reconstruction