- Euclidean Geometry: describes our world well.
- Measurements of lengths, angles, parallelism & orthogonality make sense because they are preserved by change of coordinates \( \rightarrow \) rotation+translation \( \rightarrow \) Euclidean Transformation.

For to describe projections \( \rightarrow \) projective geometry.
- Railroad Tracks are parallel in 3D space but converge when they are projected onto an image.
- All points at \( \infty \) in 3D have the same projection as observer moves.
- Since parallelism is not preserved by projection, neither are distance or angles.

Projective Geometry is an extension of Euclidean geometry that describes a larger class of transformations (not just rot+trans), i.e. perspective transformations.
Projective Geometry deals with phenomena at $\infty$.

- Euclidean coordinate in a plane $[u, v]^T$
- Obtain projective coordinate by just adding a 1 at the end: $[u, v, 1]^T$
- To make a one-to-one correspondence between Euclidean coordinate & projective coordinate make a rule: scaling by a non-zero factor in Projective coordinate is not significant $\Rightarrow [u, v, 1]^T$ & $[ku, kv, 1]^T$ represent the same point.

Space of $(n+1)$ Tuples of coordinate with the rule that proportional $(n+1)$ Tuples represent the same point is called Projective Space of dimension $n$, denoted by $\mathbb{P}^n$. 
Object space $\mathbb{P}^3$
- Image space $\mathbb{P}^2$
- Given coordinates in $\mathbb{R}^n \rightarrow$ can build coordinate in $\mathbb{P}^n$

$$[x_1, \ldots, x_n]^T \rightarrow [x_1, \ldots, x_n, 1]^T$$

Reverse:

$$[x_1, \ldots, x_{n+1}] \rightarrow \left[\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right]^T$$

- More points in $\mathbb{P}^n$ than $\mathbb{R}^n$
- Point $[x_1, \ldots, x_{n+1}]^T$ in $\mathbb{P}^n$ with $x_{n+1} \neq 0 \rightarrow$ usual point
- But if $x_{n+1} = 0 \rightarrow$ no Euclidean context

Consider the limit $\left[\frac{x_1}{\lambda}, \ldots, \frac{x_n}{\lambda}, 1\right]^T$ as $\lambda \rightarrow 0$

- Limit of a point of $\mathbb{R}^n$ going to $\infty$ in the direction of $[x_1, \ldots, x_n]^T$
- Called point at $\infty$

$$\mathbb{P}^n = \mathbb{R}^n \cup \text{points at } \infty \cup \left\{[x_1, \ldots, x_n, 0]^T \mid (x_1, \ldots, x_n) \in \mathbb{R}^n\right\}$$
points at $o$ not special – treated like all other points.

- In Projective Space Point & line have the same representation.
- X product denote meet & join
- Simple matrix operator represent geometric union of 2 points to form a line or intersection of 2 line to form a point – (no divisions)

Euclidean geometry: all lines don't meet. Apply formula to find intersection needs to dividing by zero.

Projective geometry: no need to handle special cases.
Points at $o$ are handled like ordinary points.
Pinhole model.

- Plane $R \rightarrow$ image plane on retina plane
- Point $C$ not in $R \rightarrow$ optical center

Projection $m$ of a point of space $M$ is the intersection of optical ray $(C, M)$ with retina plane.

Relation between coordinates of $M (x, y, z)^T$ and its projection $m (u, v)^T$:

$$u = \frac{x}{z}, \quad v = \frac{y}{z}$$

Point $M$ : whole incoming ray.

Optical ray of $m$ contains optical center.

$\Rightarrow$ To derive its position in 3D camera coordinate system specify another point $(x', y', z')^T$ on $[dx, dy, dz]^T$.

$\Rightarrow$ Using geometry alone cannot infer 3D depth of point from a single image.
This ambiguity is described by considering 
\[ [dx, dy, dz] \] to be the projective coordinates of an optical ray.

Choose the optical plane at \( z = 1 \)

\( m : [u, v] \) in Euclidean, but \( [u, v, 1]^T \) in 3D \( \Rightarrow \) same

\( \Rightarrow \) in projective coordinates:

\( \Rightarrow \) represent a 3D point in 3D on the optical ray as \( m \).