Multiresolution coding and wavelets

- Predictive (closed-loop) pyramids
- Open-loop ("Laplacian") pyramids
- Discrete Wavelet Transform (DWT)
- Quadrature mirror filters and conjugate quadrature filters
- Lifting and reversible wavelet transform
- Wavelet theory
- Embedded zero-tree wavelet (EZW) coding

Interpolation error coding, I

Input picture

Subsampling

Interpolator

Coder includes

Decoder

Reconstructed picture

Sample encoded in current stage

Previously coded sample
Interpolation error coding, II

Predictive pyramid, I

Input picture

Filtering

Subsampling

Filtering

Subsampling

Coder includes

Decoder

Sample encoded in current stage

Original image

Signals to be encoded
Predictive pyramid, II

Number of samples to be encoded =
\[
\left(1 + \frac{1}{N} + \frac{1}{N^2} + \ldots\right) = \frac{N}{N-1} \times \text{number of original image samples}
\]

subsampling factor

Predictive pyramid, III

original image

signals to be encoded

Bernd Girod: EE398A Image Communication I Multiresolution & Wavelets no. 5

Bernd Girod: EE398A Image Communication I Multiresolution & Wavelets no. 6
Comparison: interpolation error coding vs. pyramid

- Resolution layer #0, interpolated to original size for display

Interpolation Error Coding  Pyramid

Comparison: interpolation error coding vs. pyramid

- Resolution layer #1, interpolated to original size for display

Interpolation Error Coding  Pyramid
Comparison: interpolation error coding vs. pyramid

- Resolution layer #2, interpolated to original size for display

Interpolation Error Coding      Pyramid

Comparison: interpolation error coding vs. pyramid

- Resolution layer #3

Interpolation Error Coding      Pyramid

= (original)
Open-loop pyramid (Laplacian pyramid)

When multiresolution coding was a new idea . . .

This manuscript is okay if compared to some of the weaker papers. [. . .] however, I doubt that anyone will ever use this algorithm again.

Anonymous reviewer of Burt and Adelson’s original paper, ca. 1982
Cascaded analysis / synthesis filterbanks

Discrete Wavelet Transform

- Recursive application of a two-band filter bank to the lowpass band of the previous stage yields octave band splitting:

- Same concept can be derived from wavelet theory:
  **Discrete Wavelet Transform (DWT)**
2-d Discrete Wavelet Transform

...etc

2-d Discrete Wavelet Transform example
2-d Discrete Wavelet Transform example
2-d Discrete Wavelet Transform example

2-d Discrete Wavelet Transform example
Two-channel filterbank

\[ x(z) = \frac{1}{2} \left[ h_0(z) g_0(z) + h_1(z) g_1(z) \right] x(z) \]

\[ + \frac{1}{2} \left[ h_0(-z) g_0(z) + h_1(-z) g_1(z) \right] x(-z) \]

Aliasing cancellation if:

\[ g_0(z) = h_1(-z) \]

\[ -g_1(z) = h_0(-z) \]

Example: two-channel filter bank with perfect reconstruction

- Impulse responses, analysis filters:
  - Lowpass
  - Highpass
  \[ \begin{pmatrix} -1 & 3 & 1 & -1 \\ 4 & 2 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 4 \end{pmatrix} \]

- Impulse responses, synthesis filters:
  - Lowpass
  - Highpass
  \[ \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 3 & 1 \\ 4 & 2 & 2 & 4 \end{pmatrix} \]

"Biorthogonal 5/3 filters"
"LeGall filters"

Mandatory in JPEG2000

Frequency responses:
Classical quadrature mirror filters (QMF)

- QMFs achieve aliasing cancellation by choosing

\[ h_i(z) = h_0(-z) \]
\[ = -g_1(z) = g_0(-z) \]

[Croisier, Esteban, Galand, 1976]

- Highpass band is the mirror image of the lowpass band in the frequency domain
- Need to design only one prototype filter

Example:
16-tap QMF filterbank

Conjugate quadrature filters

- Achieve aliasing cancelation by

\[ h_0(z) = g_0(z^{-1}) \equiv f(z) \]
\[ h_i(z) = g_1(z^{-1}) = zf(-z^{-1}) \]

[Smith, Barnwell, 1986]

- Impulse responses

\[ h_0[k] = g_0[-k] = f[k] \]
\[ h_i[k] = g_1[-k] = (-1)^{k+1} f[-(k+1)] \]

- Orthonormal subband transform!
- Perfect reconstruction: find power complementary prototype filter

\[ |F(\omega)|^2 + |F(\omega \pm \pi)|^2 = 2 \]
Lifting

- **Analysis filters**
  - Even samples $x[2n]$
  - Odd samples $x[2n+1]$

- $L$ "lifting steps"  
  - First step can be interpreted as prediction of odd samples from the even samples
  
  $$\begin{align*}
  &\lambda_1 \lambda_2 \cdots \lambda_{L-1} \lambda_L \\
  &\sum \quad \sum \\
  &\text{low band } y_e \\
  &\text{high band } y_i
  \end{align*}$$

  [Sweldens 1996]

Lifting (cont.)

- **Synthesis filters**
  - Even samples $x[2n]$
  - Odd samples $x[2n+1]$

- Perfect reconstruction (biorthogonality) is directly built into lifting structure

- Powerful for both implementation and filter/wavelet design

$$\begin{align*}
  &\lambda_1 \lambda_2 \cdots \lambda_{L-1} \lambda_L \\
  &\sum \quad \sum \\
  &\text{low band } y_e \\
  &\text{high band } y_i
  \end{align*}$$
Example: lifting implementation of 5/3 filters

\[
\begin{align*}
\text{even samples } x[2n] & \rightarrow \sum & \text{low band } y_0 \\
\frac{-(1+z)}{2} & \quad \frac{1+z^{-1}}{4} \\
\text{odd samples } x[2n+1] & \rightarrow \sum & \text{high band } y_1 \\
\end{align*}
\]

Verify by considering response to unit impulse in even and odd input channel.

Reversible subband transform

- Observation: lifting operators can be nonlinear.
- Incorporate the necessary rounding into lifting operator:

\[
\begin{align*}
\text{even samples } x[2n] & \rightarrow \sum \quad \lambda_1 \quad \lambda_2 \\
\text{odd samples } x[2n+1] & \rightarrow \sum \\
\lambda_{L-1} & \quad \lambda_L \\
\end{align*}
\]

- Used in JPEG2000 as part of 5/3 biorthogonal wavelet transform
Wavelet bases

Consider Hilbert space $\mathcal{L}^2(\mathbb{R})$ of finite-energy functions $x = x(t)$.

Wavelet basis for $\mathcal{L}^2(\mathbb{R})$: family of linearly independent functions

$$\psi_n^m(t) = \sqrt{2^{-m}} \psi(2^{-m} t - n)$$

that span $\mathcal{L}^2(\mathbb{R})$. Hence any signal $x \in \mathcal{L}^2(\mathbb{R})$ can be written as

$$x = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y(m)[n] \psi_n^m$$

Multi-resolution analysis

Nested subspaces

$$\ldots \subset V^{(2)} \subset V^{(1)} \subset V^{(0)} \subset V^{(-1)} \subset V^{(-2)} \subset \ldots \subset \mathcal{L}^2(\mathbb{R})$$

Upward completeness

$$\bigcup_{n \in \mathbb{Z}} V^{(n)} = \mathcal{L}^2(\mathbb{R})$$

Downward completeness

$$\bigcap_{n \in \mathbb{Z}} V^{(n)} = \{0\}$$

Self-similarity

$$x(t) \in V^{(0)} \iff x\left(2^{-m} t\right) \in V^{(m)}$$

Translation invariance

$$x(t) \in V^{(0)} \iff x(t-n) \in V^{(0)} \text{ for all } n \in \mathbb{Z}$$

There exists a "scaling function" $\varphi(t)$ with integer translates $\varphi_n(t) = \varphi(t-n)$ such that $\{\varphi_n\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for $V^{(0)}$.
Multiresolution Fourier analysis

\[
\text{span} \left\{ \phi^{(p-3)}_n \right\}_n = V^{(p-3)}
\]

\[
\text{span} \left\{ \phi^{(p-2)}_n \right\}_n = V^{(p-2)}
\]

\[
\text{span} \left\{ \phi^{(p-1)}_n \right\}_n = V^{(p-1)}
\]

\[
\text{span} \left\{ \phi^{(p)}_n \right\}_n = V^{(p)}
\]

\[
\text{span} \left\{ \psi^{(p-3)}_n \right\}_n = W^{(p-3)}
\]

\[
\text{span} \left\{ \psi^{(p-2)}_n \right\}_n = W^{(p-2)}
\]

\[
\text{span} \left\{ \psi^{(p-1)}_n \right\}_n = W^{(p-1)}
\]

\[
\text{span} \left\{ \psi^{(p)}_n \right\}_n = W^{(p)}
\]

Relation to subband filters

Since \(V^{(0)} \subseteq V^{(-1)}\), recursive definition of scaling function

\[
\varphi(t) = \sum_{n=-\infty}^{\infty} g_0[n] \varphi_n^{-1}(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n)
\]

linear combination of scaling functions in \(V^{(-1)}\)

Orthonormality

\[
\delta[n] = \left\langle \varphi^{(0)}_n, \varphi^{(0)}_n \right\rangle = \int_{-\infty}^{\infty} \left( \sum_i g_0[i] \varphi_i^{-1}(t) \sum_j g_0[j] \varphi_j^{-1}(2t) \right) dt
\]

\[
= \sum_{i,j} g_0[i] g_0[j-2n] \left\langle \varphi_i^{-1}, \varphi_j^{-1} \right\rangle = \sum_{i,j} g_0[i] g_0[i] g_0[j] \text{ unit norm and orthogonal to its 2-translates; corresponds to synthesis lowpass filter of orthonormal subband transform}.
\]
Wavelets from scaling functions

\( W^{(p)} \) is orthogonal complement of \( V^{(p)} \) in \( V^{(p-1)} \)

\[ W^{(p)} \perp V^{(p)} \quad \text{and} \quad W^{(p)} \cup V^{(p)} = V^{(p-1)} \]

Orthonormal wavelet basis \( \{ \psi_n^{(0)} \} \) for \( W^{(0)} \subset V^{(-1)} \)

\[ \psi(t) = \sum_{n=-\infty}^{\infty} g_1[n] \phi_n^{(-1)}(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \phi_n(2t-n) \]

Using conjugate quadrature high-pass synthesis filter

\[ g_1[n] = (-1)^{n+1} g_0[-(n-1)] \]

The mutually orthonormal functions, \( \{ \psi_n^{(0)} \}_{n \in \mathbb{Z}} \) and \( \{ \phi_n^{(0)} \}_{n \in \mathbb{Z}} \), together span \( V^{(-1)} \).

Easy to extend to dilated versions of \( \psi(t) \) to construct orthonormal wavelet basis \( \{ \psi_n^{(m)} \}_{n \in \mathbb{Z}} \) for \( L^2(\mathbb{R}) \).

Calculating wavelet coefficients for a continuous signal

- Signal synthesis by discrete filter bank

Suppose continuous signal \( x^{(0)}(t) = \sum_{n \in \mathbb{Z}} y_n^{(0)}[n] \varphi(t-n) = \sum_{n \in \mathbb{Z}} y_n^{(0)}[n] \varphi_n^{(0)} \in V^{(0)} \)

Write as superposition of \( x^{(1)}(t) \in V^{(1)} \) and \( w_1^{(1)}(t) \in W^{(1)} \)

\[ x^{(0)}(t) = \sum_{i \in \mathbb{Z}} y_0^{(i)}[i] \phi_n^{(i)} + \sum_{j \in \mathbb{Z}} y_1^{(j)}[j] \varphi_n^{(j)} \]

\[ = \sum_{n \in \mathbb{Z}} \varphi_n^{(0)} \left( \sum_{i \in \mathbb{Z}} y_0^{(i)}[n] g_0[n-2i] + \sum_{j \in \mathbb{Z}} y_1^{(j)}[j] g_1[n-2j] \right) \]

- Signal analysis by analysis filters \( h_0[k], h_1[k] \)
- Discrete wavelet transform
Discrete Wavelet Transform

Different wavelets

Haar
2/2 coeffs.

Daubechies
8/8

Symlets
8/8

Cohen-Daubechies-Feauveau
17/11

[Gonzalez, Woods, 2001]
Daubechies orthonormal 8-tap filters

$h_0(n)$

$g_0(n)$

$0 \quad 2 \quad 4 \quad 6 \quad 8$

$n$

$0 \quad 2 \quad 4 \quad 6 \quad 8$

$n$

8-tap Symlets

$h_0(n) = h_0(-n)$

$g_0(n) = h_0(n)$

$0 \quad 2 \quad 4 \quad 6 \quad 8$

$n$

$0 \quad 2 \quad 4 \quad 6 \quad 8$

$n$

[Source: Gonzalez, Woods, 2001]
Biorthogonal Cohen-Daubechies-Feauveau 17/11 wavelets

Wavelet compression results

<table>
<thead>
<tr>
<th>Original 512x512 8bpp</th>
<th>0.074 bpp</th>
<th>Error images</th>
<th>0.048 bpp</th>
<th>enlarged</th>
</tr>
</thead>
</table>

[Gonzalez, Woods, 2001]
Embedded zero-tree wavelet algorithm

- Idea: Conditional coding of all descendants (incl. children)
- Coefficient magnitude > threshold: significant coefficients
- Four cases
  - ZTR: zero-tree, coefficient and all descendants are not significant
  - IZ: coefficient is not significant, but some descendants are significant
  - POS: positive significant
  - NEG: negative significant

Embedded zero-tree wavelet algorithm (cont.)

- For the highest bands, ZTR and IZ symbols are merged into one symbol Z
- Successive approximation quantization and encoding
  - Initial „dominant“ pass
    - Set initial threshold T, determine significant coefficients
    - Arithmetic coding of symbols ZTR, IZ, POS, NEG
  - Subordinate pass
    - Refine magnitude of all coefficients found significant so far by one bit (subdivide magnitude bin by two)
    - Arithmetic coding of sequence of zeros and ones.
  - Repeat dominant pass
    - Omit previously found significant coefficients
    - Decrease threshold by factor of 2, determine new significant coefficients
    - Arithmetic coding of symbols ZTR, IZ, POS, NEG
  - Repeat subordinate and dominate passes, until bit budget is exhausted.
Embedded zero-tree wavelet algorithm (cont.)

- Decoding: bitstream can be truncated to yield a coarser approximation: “embedded” representation

Summary: multiresolution and subband coding

- Resolution pyramids with subsampling 2:1 horizontally and vertically
- Predictive pyramids: quantization error feedback („closed loop“)
- Transform pyramids: no quantization error feedback („open loop“)
- Pyramids: overcomplete representation of the image
- Critically sampled subband decomposition: number of samples not increased
- Discrete Wavelet Transform = cascaded dyadic subband splits
- Quadrature mirror filters and conjugate quadrature filters: aliasing cancellation
- Lifting: powerful for implementation and wavelet construction
- Lifting allows reversible wavelet transform
- Zero-trees: exploit statistical dependencies across subbands