Quantization

- Assume a coding image intensity value.
- f pixels can be assigned values $0 \rightarrow 255$, so 8 bits are used to code each pixel.

![Diagram of uniform quantizer]

- There are 4 bins in this uniform quantizer.
Quantization

- Non-uniform pdf:

Generalization of Quantization Problem

- $f_{\text{min}} \leq f \leq f_{\text{max}}$
- $J = \text{number of reconstruction levels}$
- $P(f) = \text{pdf for } f$
Generalization of Quantization Problem

- Choose $r_i$, $d_i$ so that they minimize:

\[
\epsilon = E \left[ (f - \hat{f})^2 \right]
\]

\[
\epsilon = \int_{f_{\min}}^{f_{\max}} (f - \hat{f})^2 p(f) df
\]

\[
\epsilon = \sum_{j=0}^{J-1} \int_{d_j}^{d_{j+1}} (f - r_j)^2 p(f) df
\]

\[
\frac{\partial \epsilon}{\partial r_j} = 0 \rightarrow r_j = 2d_j - r_{j-1}
\]

\[
\frac{\partial \epsilon}{\partial d_j} = 0 \rightarrow d_{j+1} \int f \cdot p(f) df - d_j \int p(f) df
\]
Approach for Avoiding Non-Uniform Quantization

- $f$ has a non-uniform pdf:

```
+----------------+          +----------------+
| nonlinearity   |          | Uniform Quantizer |
+----------------+          +----------------+
          ^            ^
          g            g
```

$\hat{f}$
Vector Quantization

- $f$ ranges from 0 → 255.
- There is a level assignment for 2 pixels at a time, one from $f_1$ and one from $f_2$.

\[ \mathcal{E} = \iint_{f_1, f_2} \left[ (f_1 - \hat{f}_1)^2 + (f_2 - \hat{f}_2)^2 \right] p(f_1, f_2) \, df_1 \, df_2 \]

Scalar quantization

Vector quantization "dimension 2"
Vector Quantization

\( f_2 \) (weight)

\[ \begin{array}{c}
\text{ } \\
\ \ \\
\ \ \\
\text{ } \\
\end{array} \]

\( f_1 \) (height)

\[ \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \]

\[ \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
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\end{array} \]

\[ \begin{array}{c}
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\[ \begin{array}{c}
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\end{array} \]
Entropy Coding, Codeword Design, Bit Allocation

- Shorter codewords are assigned to more probable symbols.
Entropy Coding, Codeword Design, Bit Allocation

- **Example**
  - $r_1 \rightarrow 0$
  - $r_2 \rightarrow 1$
  - $r_3 \rightarrow 10$
  - $r_4 \rightarrow 11$

- **4 symbols**

- Not uniquely decodable

- **Problem** One codeword is a prefix of others.

- **Solution** Use a “prefix coder”, where no codeword is a prefix of any other codeword $\Rightarrow$ unique decodability
Entropy Coding

• **Goal:** Design variable length codewords such that:
  - the average bit rate is minimized
  - the codewords are uniquely decodable

\[
\text{Entropy } H = \sum_{i=1}^{L} p_i \log_2 p_i
\]

\[p_i = "\text{probability of symbol } i"\]

• Entropy is the average amount of information in a message.
• 2 symbols with \( p_1 = 1 \) and \( p_2 = 0 \) \(\implies H = 0 \) \(\implies\) no information
Entropy Coding

- 2 symbols with $p_1 = 99\%$ and $p_2 = 1\%$

\[
\begin{align*}
\text{a}_1 & \rightarrow \text{a}_1 \text{ a}_1 \text{ a}_1 \text{ a}_2 \ldots \ldots \text{a}_2 \ldots \text{a}_1 \\
\text{a}_2 & \rightarrow \text{a}_1 \text{ a}_1 \text{ a}_2 \text{ a}_1 \text{ a}_2 \text{ a}_2 \text{ a}_1 
\end{align*}
\]

- 2 symbols with $p_1 = 50\%$ and $p_2 = 50\%$

\[
\begin{align*}
\text{a}_1 & \rightarrow \text{a}_1 \text{ a}_1 \text{ a}_2 \text{ a}_1 \text{ a}_2 \text{ a}_2 \text{ a}_1 \\
\text{a}_2 & \rightarrow \text{a}_1 \text{ a}_1 \text{ a}_2 \text{ a}_1 \text{ a}_2 \text{ a}_2 \text{ a}_1 
\end{align*}
\]
Entropy Coding

- Source with 2 letters with probabilities $p$ and $(1-p)$

- Source with $L$ letters

\[ 0 \leq H \leq \log_2 L \]
\[ \sum_{i=1}^{L} p_i = 1 \]