EECS 291E, Spring 2020:
Hybrid and Intelligent Control Systems Design
Homework 1: Modeling Hybrid Systems

Assigned January 30th, Due Feburary 10th, 5 pm.

Problem 1

Consider the discontinuous differential equation

$$\dot{x}_1 = -\text{sgn}(x_1) + 2\text{sgn}(x_2) \quad (1)$$

$$\dot{x}_2 = -2\text{sgn}(x_1) - \text{sgn}(x_2) \quad (2)$$

where $x(0) \neq (0, 0)$, and

$$\text{sgn}(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ -1 & \text{if } z < 0 \end{cases} \quad (3)$$

This system defines a hybrid automaton with four discrete modes having invariants corresponding to the four quadrants.

A) Specify a non-blocking and deterministic hybrid automaton modeling the system.

B) Does H accept Zeno executions for every initial state?

Problem 2

Consider three balls with unit masses and suppose that they are touching at time $t = 0$. The initial velocity of ball 1 is $w_1(0) = 1$ and balls 2 and 3 are at rest. Assume that the impact is a sequence of simple inelastic impacts. The first inelastic collision occurs between balls 1 and 2, resulting in $w_1(0+) = w_2(0+) = 0.5$ and $w_3(0+) = 0$. Since $w_2(0+) > w_3(0+)$, ball 2 hits ball 3 instantaneously giving $w_1(0++) = 0.5$ and $w_2(0++) = w_3(0++) = 0.25$. Now $w_1(0++) > w_2(0++)$ so ball 1 hits ball 2 again resulting in a new inelastic collision. This leads to an infinite sequence of collisions.

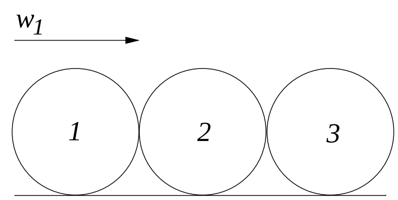

Figure 1: Three balls, for problem 2.

A) Model the inelastic collisions of the three ball system as a hybrid automaton with a single discrete mode and three continuous variables (w_1, w_2, w_3) representing the velocities of the balls.

B) Show that H accepts a Zeno execution corresponding to the sequence of collisions described above.

Problem 3

The steam boiler [1] consists of a tank containing water and a heating element that causes the water to boil and escape as steam. The water level in the boiler is denoted by w, and, for the sake of simplicity, consider only $w > 0$. The water is replenished by two pumps which at time t pump water into the boiler at rates $u_1(t)$ and $u_2(t)$ respectively. The water boils off at a rate r with d a variable that controls the rate of evaporation: $\dot{r} = d$. It is assumed the value of d at any given time is unknown, yet it is known to lie within given bounds and cannot drive r past that variable's bounds either. At every time t, pump i can be either be on ($u_i(t) = P_i$) or off ($u_i(t) = 0$). There is a delay T_i between the time pump i is ordered to switch on and the time u_i switches to P_i. There is no delay when the pumps are switched off.

The requirement is that the pumps are switched on and off so that the water level remains between two values M_1 and M_2.

A) Derive a deterministic, non-blocking, hybrid automaton for the steam boiler for appropriately chosen initial states.

B) Simulate the system for parameters: $P_i = 2.5$ and $T_i = 5$ (for $i = 1, 2$), $r \in [0, 4]$, $d \in [0, 0.5]$, $M_1 = 1, M_2 = 20$. Using your intuition, can you devise a pumping strategy that keeps the system within the allowable bounds, for a possible worst case disturbance and appropriately chosen initial states?

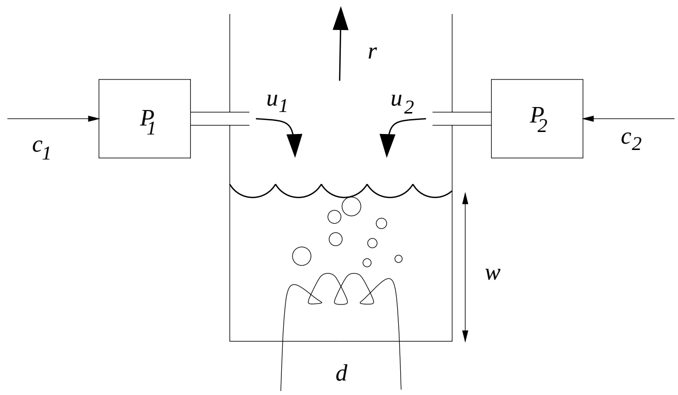

Figure 2: Steam Boiler, for problem 3.

Problem 4

Consider the inverted pendulum system in the paper "*Qualitative Modeling and Heterogeneous Control of Global System Behavior*" [2]. You can access this paper at the author's website:

`https://web.eecs.umich.edu/~kuipers/research/pubs/Kuipers-hscc-02.html`

Using the model and control scheme outlined in Section 3.5 of the paper, implement a controlled inverted pendulum simulation. Include some simulations to show how the pendulum swings up for a few initial conditions. What are possible benefits of this hybrid control scheme against a continuous nonlinear control scheme?

Problem 5

Consider the water tank hybrid automaton given in Figure 4 of Lecture Notes 3 (of Professor Claire Tomlin, 2018)

A) Show that if $w > \max\{v_1, v_2\}$ then the set $Q \times \{x \in \mathbb{R}^2 : x_1 \geq r_1 \text{ AND } x_2 \geq r_2\}$ is invariant.

B) Assume that $\max\{v_1, v_2\} < w < v_1 + v_2$, so that the water tank hybrid automaton is Zeno.

   Temporal regularization of this system refers to the situation in which there is a delay $\epsilon > 0$ between the time the inflow is commanded to switch from one tank to the other, and the time the switch actually takes place.

   Derive a new hybrid automaton for the water tank which incorporates this regularization, and show that this regularized automaton accepts a unique non-Zeno execution for each initial state.

References

[1] J.-R. Abrial, E. Börger, and H. Langmaack. The steam-boiler case study project, an introduction. In J.-R. Abrial, E. Börger, and H. Langmaack, editors, *Formal Methods for Industrial Applications: Specifying and Programming the Steam Boiler Control*, number 1165 in LNCS. Springer Verlag, 1996.

[2] Benjamin Kuipers and Subramanian Ramamoorthy. Qualitative modeling and heterogeneous control of global system behavior. In *International Workshop on Hybrid Systems: Computation and Control*, pages 294–307. Springer, 2002.