EE291E/ME290A Lecture Notes 12. Controllers for Safety for Continuous Systems

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The material in this lecture is based on the presentations in [1, 2, 3, 4, 5, 6].

In the last lectures we have discussed optimal control and game theory for continuous systems: we highlighted mathematical tools (calculus of variations and dynamic programming) that may be used to solve these problems. In this lecture, we show how these methods may be applied to the problem of solving the reachability problem and synthesizing controllers for continuous systems:

$$\dot{x}(t) = f(x(t), u(t), d(t)) \tag{1}$$

where $x \in X$ where $X = \mathbb{R}^n$, $u \in U$ is the set of control inputs and $d \in D$ is the set of disturbance inputs, where U and D are convex and compact subsets of $\mathbb{R}^u$ and $\mathbb{R}^d$ respectively; f is a vector field, assumed to be globally Lipschitz in x and continuous in u and d; and the initial state $x(0) \in$ Init where Init $\subseteq \mathbb{R}^n$. We will use $\mathcal{U}$ to denote the set of piecewise continuous functions from $\mathbb{R}$ to U and $\mathcal{D}$ to denote the set of piecewise continuous functions from $\mathbb{R}$ to D.

Consider a set of states $F \subseteq \mathbb{R}^n$. Try to establish the maximal controlled invariant subset of F, i.e. the largest set of initial states for which there exists a controller that manages to keep all executions inside F. Somewhat informally this set can be characterized as:

$$W^* = \{x_0 \in \mathbb{R}^n \ : \ \exists u \in \mathcal{U} \ \forall d \in \mathcal{D}, \ \Box F(x_0, u, d)\}$$

To eliminate technical complications we assume that:

Assumption 1 *There exists a continuously differentiable function $l : \mathbb{R}^n \to \mathbb{R}$ such that:*

$$
\begin{aligned}
l(x) &> 0 &&if && x \in F^o \\
l(x) &= 0 &&if && x \in \partial F \\
l(x) &< 0 &&if && x \in F^c \\
l(x) &= 0 &&\Rightarrow && \tfrac{\partial l}{\partial x}(x) \neq 0
\end{aligned}
$$

The assumption implies that F is a closed set with non-empty interior, whose boundary is a $n-1$ dimensional manifold.

1   Dynamic Programming Solution

To apply the optimal control and differential game tools introduced in the previous lecture let $t_f = 0$, consider an arbitrary $t \leq 0$ and introduce the value function:

$$\hat{J}(x,t) = \max_{u \in \mathcal{U}_{[t,0]}} \min_{d \in \mathcal{D}_{[t,0]}} l(x(0))$$

Notice that this control problem involves no Lagrangian ($L \equiv 0$), just a terminal cost. The game obtained in this setting falls in the class of *pursuit-evasion* games [7]. The (u,d) obtained by the above optimal control problem is a Stackelberg equilibrium for the game between control and disturbance, with the control playing the role of the leader.

$\hat{J}$ can be computed using the dynamic programming tools discussed in the last two lectures. Introduce a Hamiltonian:

$$\begin{aligned} H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^u \times \mathbb{R}^d &\longrightarrow \mathbb{R} \\ (x,p,u,d) &\longmapsto p^T f(x,u,d) \end{aligned}$$

Consider the optimal Hamiltonian:

$$H^*(x,p) = \max_{u \in U} \min_{d \in D} H(x,p,u,d)$$

Notice again that the minimization over u and d is pointwise, as opposed to over functions of time. Then, if $\hat{J}$ is continuously differentiable it satisfies:

$$\begin{aligned} \tfrac{\partial \hat{J}}{\partial t}(x,t) &= -H^*\left(x, \tfrac{\partial \hat{J}}{\partial x}(x,t)\right) \\ \hat{J}(x,0) &= l(x) \end{aligned} \tag{2}$$

Notice that the evolution of the partial differential equation is "backwards" in time.

Consider the set:

$$\widehat{W}_t = \{x_0 \in X \ : \ \hat{J}(x_0,t) \geq 0\}$$

This is the set of all states for which starting at $x(t) = x_0$, there exists a controller for u such that for all disturbance trajectories $d \in \mathcal{D}_{[t,0]}$, $l(x(0)) \geq 0$ or, in other words, $x(0) \in F$. This is not quite what we need yet. We would like the set of all states for which there exists a u such that for all d and for all $t' \in [t,0]$, $x(t') \in F$. This excludes points in $\widehat{W}_t$ which leave F at some point in $[t,0]$ but re-enter it before time 0. This requirement can be encoded either after the computation of $\hat{J}$, or by modifying the Hamilton-Jacobi equation:

$$\begin{aligned} \tfrac{\partial J}{\partial t}(x,t) &= -\min\left\{0, H^*\left(x, \tfrac{\partial J}{\partial x}(x,t)\right)\right\} \\ J(x,0) &= l(x) \end{aligned} \tag{3}$$

Compare this with the discrete Hamilton-Jacobi equation from Lecture 7:

$$\begin{aligned} J(q,0) &= \begin{cases} 1 & q \in F \\ 0 & q \in F^c \end{cases} \\ J(q,i-1) - J(q,i) &= \min\{0, \max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2}[\min_{q' \in R(q,\sigma_1,\sigma_2)} J(q',i) - J(q,i)]\} \end{aligned} \tag{4}$$

$R(q,\sigma_1,\sigma_2)$ essentially implements the spatial partial derivative of J along the dynamics of the system. The innermost minimization is not needed in the continuous case, as the continuous system is "deterministic".

As before, we seek a stationary solution to this equation. Assume that as $t \to -\infty$, $J(x,t)$ converges to a continuously differentiable function $J^* : X \to \mathbb{R}$.

Proposition 2 *The set $W^* = \{x \in X \; : \; J^*(x) \geq 0\}$ is the largest controlled invariant set contained in F.*

The solution to the partial differential equation also leads to a least restrictive controller that renders W^* invariant. Consider:

$$g(x) = \begin{cases} \left\{ u \in U \; : \; \min_{d \in D} \left(\frac{\partial J^*(x)}{\partial x} \right)^T f(x, u, d) \geq 0 \right\} & \text{if } x \in \partial W^* \\ U & \text{if } x \in (W^*)^\circ \cup (W^*)^c \end{cases} \tag{5}$$

2 Geometric interpretation

For an arbitrary time $t \leq 0$ define:

$$W_t = \{x \in X \; : \; J(x, t) \geq 0\}$$

Consider an $x \in \partial W_t$ and assume that:

$$H^* \left(x, \frac{\partial J}{\partial x}(x, t) \right) < 0$$

$$\Leftrightarrow \quad \max_{u \in U} \min_{d \in D} H \left(x, \frac{\partial J}{\partial x}(x, t), u, d \right) < 0$$

$$\Leftrightarrow \quad \max_{u \in U} \min_{d \in D} \frac{\partial J}{\partial x}(x, t) f(x, u, d) < 0$$

$$\Leftrightarrow \quad \forall u \in U \; \exists d \in D \text{ such that } \frac{\partial J}{\partial x}(x, t) f(x, u, d) < 0$$

But $\frac{\partial J}{\partial x}(x, t)$ is the normal to the boundary of W_t at x, pointing inside W_t. Moreover, $\frac{\partial J}{\partial x}(x, t) f(x, u, d)$ is the inner product between this normal and the vector $f(x, u, d)$. Let θ be the angle between $\frac{\partial J}{\partial x}(x, t)$ and $f(x, u, d)$. Then:

$$\frac{\partial J}{\partial x}(x, t) f(x, u, d) > 0 \quad \text{if} \quad \theta < \pi/2$$

$$\frac{\partial J}{\partial x}(x, t) f(x, u, d) = 0 \quad \text{if} \quad \theta = \pi/2$$

$$\frac{\partial J}{\partial x}(x, t) f(x, u, d) < 0 \quad \text{if} \quad \theta > \pi/2$$

Therefore, the above statement is equivalent to:

> for all $u \in U$ there exists $d \in D$ such that the normal to ∂W_t at x pointing towards the interior of W_t makes an angle greater than $\pi/2$ with $f(x, u, d)$,

or, equivalently:

> for all $u \in U$ there exists $d \in D$ such that $f(x, u, d)$ points outside W_t.

These are points at which whatever u does d can force them to leave the set W_t instantaneously. Notice that the order of the quantifiers in the above expression implies that d may depend on u, in

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addition to x and t. The part of the boundary of W_t where $H^* < 0$ is known as the "usable part" in the pursuit-evasion game literature.

Returning to the Hamilton-Jacobi equation, we see that for these points:

$$\frac{\partial J}{\partial t}(x,t) = -\min\left\{0, H^*\left(x, \frac{\partial J}{\partial x}(x,t)\right)\right\}$$
$$= -H^*\left(x, \frac{\partial J}{\partial x}(x,t)\right)$$
$$> 0$$

Therefore, as t decreases, J also decreases. For these points on the boundary of W_t, J becomes negative instantaneously, and they "fall out of" W_t.

What if $H^* \geq 0$? A similar argument shows that in this case:

> there exists $u \in U$ such that for all $d \in D$ the normal to ∂W_t at x pointing towards the interior of W_t makes an angle at most $\pi/2$ with $f(x,u,d)$,

or, equivalently:

> there exists $u \in U$ such that for all $d \in D$, $f(x,u,d)$ either points inside W_t or is tangent to ∂W_t.

These are points for which there exists a choice of u that for all d forces the state to remain in W_t. Notice that the order of the quantifiers implies that u may only depend x and t, and not d. For these points:

$$\frac{\partial J}{\partial t}(x,t) = -\min\left\{0, H^*\left(x, \frac{\partial J}{\partial x}(x,t)\right)\right\}$$
$$= 0$$

Therefore, as t decreases, J remains constant. These are points that want to move towards the interior of W_t. The role of the outermost minimum is to ensure that the value of J does not increase for these points, so that W_t does not grow. This is to prevent states that have been labeled as unsafe (can reach F^c) from being relabeled as safe later on.

3 Formulation as a Variational Inequality

For problems with time-varying dynamics, targets and constraints, the approach proposed in [8] requires augmenting the state space with an additional dimension accounting for time; one can then transform time-dependence into state-dependence and apply the above described methods to solve the fixed problem in the space-time state space. In more recent work [5] and [6], we have developed an extension of the Hamilton-Jacobi reach-avoid formulation to the case in which the target set, the state constraint set, and dynamics are allowed to be time-varying, enabling computation of the reach-avoid set at no significant additional cost relative to the time-invariant case. To this end, in this paper we have formulated a "double-obstacle HJI variational inequality", and proven that the zero sublevel set of its viscosity solution characterizes the desired reach-avoid set.

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4 Examples: Two-Aircraft Collision Avoidance (no mode switching)

4.1 Angular velocities as control inputs

Consider the relative model of two aircraft (From Lecture 7, with no mode switching) for the case in which the linear velocities of both aircraft are fixed, $v_1, v_2 \in \mathbb{R}$, and the control inputs of the aircraft are the angular velocities, $u = \omega_1$ and $d = \omega_2$:

$$\begin{aligned}
\dot{x}_r &= -v_1 + v_2 \cos \psi_r + u y_r \\
\dot{y}_r &= v_2 \sin \psi_r - u x_r \\
\dot{\psi}_r &= d - u
\end{aligned} \tag{6}$$

with state variables $(x_r, y_r, \psi_r) \in \mathbb{R}^2 \times [-\pi, \pi)$ and control and disturbance inputs $u \in U = [\underline{\omega}_1, \overline{\omega}_1] \subset \mathbb{R}$, $d \in D = [\underline{\omega}_2, \overline{\omega}_2] \subset \mathbb{R}$. Without loss of generality (we scale the coefficients of u and d if this is not met), assume that $\underline{\omega}_i = -1$ and $\overline{\omega}_i = 1$, for $i = 1, 2$.

The set $F^c = G$ is defined in the relative frame:

$$G = \{(x_r, y_r) \in \mathbb{R}^2, \psi_r \in [-\pi, \pi) \mid x_r^2 + y_r^2 \leq 5^2\} \tag{7}$$

and the unsafe (or capture) set is defined as the interior of G

$$G^\circ = \{(x_r, y_r) \in \mathbb{R}^2, \psi_r \in [-\pi, \pi) \mid x_r^2 + y_r^2 < 5^2\} \tag{8}$$

which is a 5-mile-radius cylindrical block in the (x_r, y_r, ψ_r) space denoting the protected zone in the relative frame. The function $l(x)$ is defined as

$$l(x) = x_r^2 + y_r^2 - 5^2 \tag{9}$$

The optimal Hamiltonian is

$$H^*(x, p) = \max_{u \in U} \min_{d \in D} [-p_1 v_1 + p_1 v_2 \cos \psi_r + p_2 v_2 \sin \psi_r + (p_1 y_r - p_2 x_r - p_3) u + p_3 d] \tag{10}$$

Defining the *switching functions* $s_1(t)$ and $s_2(t)$, as

$$\begin{aligned}
s_1(t) &= p_1(t) y_r(t) - p_2(t) x_r(t) - p_3(t) \\
s_2(t) &= p_3(t)
\end{aligned} \tag{11}$$

the optimal control and disturbance u^* and d^* exist when $s_1 \neq 0$ and $s_2 \neq 0$ and are calculated as

$$\begin{aligned}
u^* &= sgn(s_1) \\
d^* &= -sgn(s_2)
\end{aligned} \tag{12}$$

The equations for $\dot{p}$ are obtained through Hamilton's equations and are

$$\begin{aligned}
\dot{p}_1 &= u^* p_2 \\
\dot{p}_2 &= -u^* p_1 \\
\dot{p}_3 &= p_1 v_2 \sin \psi_r - p_2 v_2 \cos \psi_r
\end{aligned} \tag{13}$$

with $p(0) = (x_r, y_r, 0)^T = \nu$, the outward pointing normal to ∂G at any point (x_r, y_r, ψ_r) on ∂G.

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The usable part UP of ∂G is calculated with $\nu = (x_r, y_r, 0)^T$:

$$UP = \{(x_r, y_r, \psi_r) \in \partial G \mid -v_1 x_r + v_2(x_r \cos \psi_r + y_r \sin \psi_r) < 0\} \tag{14}$$

with boundary

$$\{(x_r, y_r, \psi_r) \in \partial G \mid -v_1 x_r + v_2(x_r \cos \psi_r + y_r \sin \psi_r) = 0\} \tag{15}$$

To solve for $p^*(t)$ and $x^*(t)$ for $t < 0$, we must first determine $u^*(0)$ and $d^*(0)$. Equations (12) are not defined at $t = 0$, since $s_1 = s_2 = 0$ on ∂G, giving rise to "abnormal extremals" [9] (meaning that the optimal Hamiltonian loses dependence on u and d at these points). Analogously to [7] (pages 442-443), we use an indirect method to calculate $u^*(0)$ and $d^*(0)$: at any point (x_r, y_r, ψ_r) on ∂G, the derivatives of the switching functions s_1 and s_2 are

$$\dot{s}_1 = y_r v_1 \tag{16}$$
$$\dot{s}_2 = x_r v_2 \sin \psi_r - y_r v_2 \cos \psi_r \tag{17}$$

For points $(x_r, y_r, \psi_r) \in \partial G$ such that $\psi_r \in (0, \pi)$ it is straightforward to show that $\dot{s}_1 > 0$ and $\dot{s}_2 > 0$, meaning that for values of t slightly less than 0, $s_1 < 0$ and $s_2 < 0$. Thus for this range of points along ∂G, $u^*(0) = -1$ and $d^*(0) = 1$. These values for u^* and d^* remain valid for $t < 0$ as long as $s_1(t) < 0$ and $s_2(t) < 0$. When $s_1(t) = 0$ and $s_2(t) = 0$, the optimal solution (u^*, d^*) switches and the computation of the boundary continues with the new values of u^* and d^*, thus introducing "kinks" into the boundary. These points correspond to loss of smoothness in the Hamilton-Jacobi equation. Figure 1 displays the resulting boundary $\{x \in X \mid J^*(x, t) = 0\}$, computed by solving the Hamilton-Jacobi equation locally using Hamilton's equations, The global solution to the Hamilton-Jacobi equation is shown in Figure 2, using the Hamilton-Jacobi equation solver of [3].

4.2 Linear velocities as control inputs

In modes in which the aircraft do not change their heading, only their linear velocities, the airspeeds vary over specified ranges: $u \in U = [\underline{v}_1, \overline{v}_1] \subset \mathbb{R}^+$, $d \in D = [\underline{v}_2, \overline{v}_2] \subset \mathbb{R}^+$, and model reduces to

$$\dot{x}_r = -u + d \cos \psi_r$$
$$\dot{y}_r = d \sin \psi_r \tag{18}$$
$$\dot{\psi}_r = 0$$

Using the same set G as in the previous subsection, we can form the optimal Hamiltonian for this problem, and solve Hamilton's equations for $x(t)$ and $p(t)$ to compute the boundary of the reachable set, as before. This is for you to do in Problem 1 of Homework 3.

5 Example: Aerodynamic Envelope Protection

This example was first inspired by the work of Charlie Hynes at NASA Ames [10], who recognized the problems in designing 'discrete' flight management systems for the continuous dynamics of an aircraft. At Ames, we solved the following 'aerodynamic envelope protection example', which seeks the maximal controlled invariant set contained within a given flight envelope. Subsequently, the guidance and control group at Honeywell Technology Center became interested in this work in the
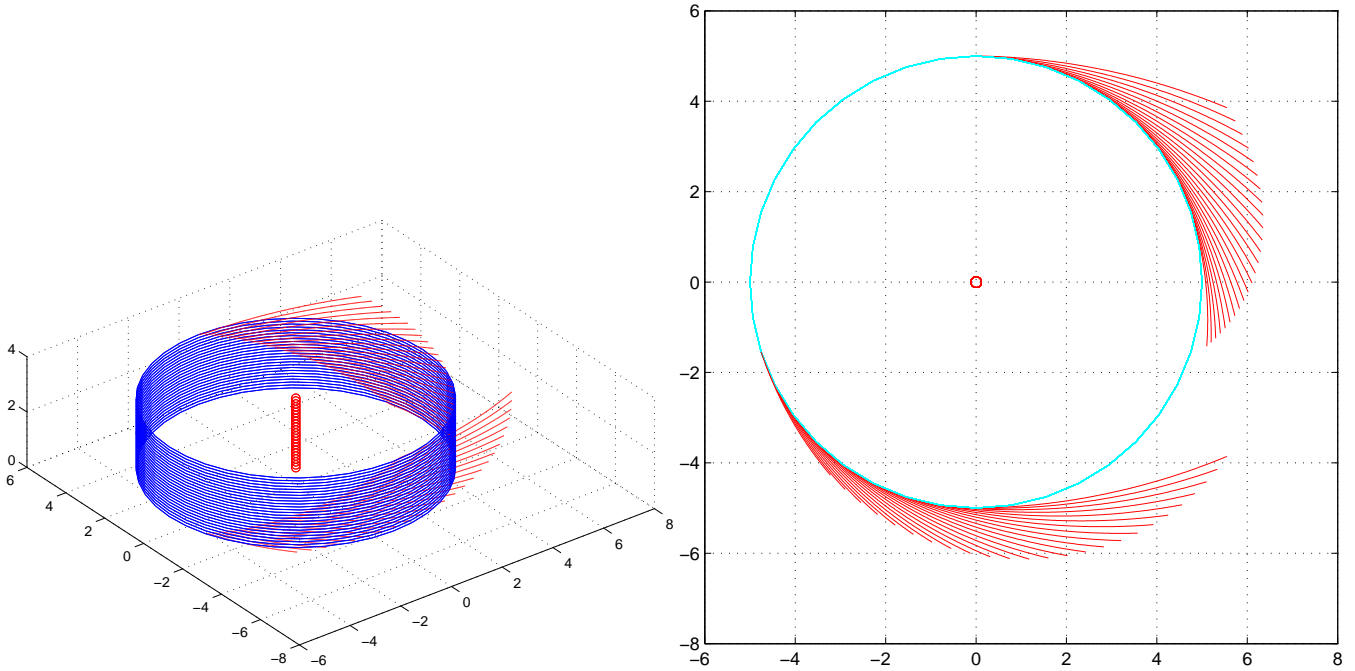
Figure 1: The set $G = \{(x_r, y_r), \psi_r \in (pi/4, \pi) \mid x_r^2 + y_r^2 \leq 5^2\}$ (cylinder) and the local representation of the boundary of the set $\{x \in X \mid J^*(x, t) = 0\}$, for fixed $t < 0$, using the solution of Hamilton's equations from the boundary of the usable part on G. The picture on the right is a top view of the one on the left.
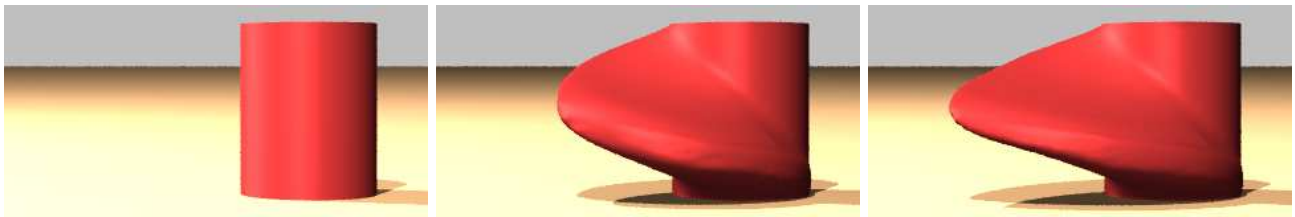

Figure 2: Level set solution to the collision avoidance example. Axes are (x_r, y_r, ψ_r), three plots are displayed to show the growth of the set, the set in the right subplot illustrates the fixed point, and it encloses all states which could eventually lead to collision under disturbance action.
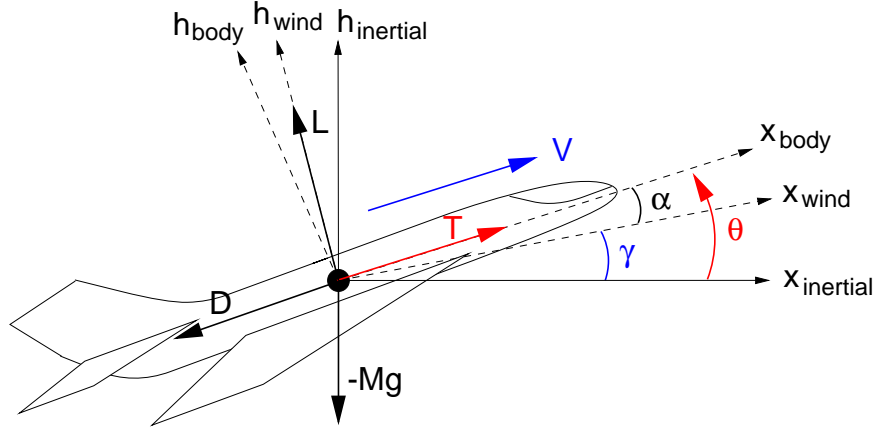
Figure 3: A planar aircraft in flight with attached axes about its center of mass.

context of the 'flight mode switching' problem (as outlined in Lecture 1), and we extended the continuous problem below to the multiple mode hybrid system. The example presented below is based on our presentations in [11, 1].

We consider a nonlinear model of the longitudinal axis dynamics of a conventional take-off and landing (CTOL) aircraft in normal aerodynamic flight in still air [12] shown in Figure 3. The horizontal and vertical axes are respectively the $(x_{inertial}, h_{inertial})$ (denoted x, h) axes and the *pitch angle* θ is the angle made by the aircraft body axis, x_{body} with the x axis. The *flight path angle* γ and the *angle of attack* α are defined as: $\gamma = \tan^{-1}(\frac{\dot{h}}{\dot{x}})$, $\alpha = \theta - \gamma$. Expressions for the lift (L) and drag (D) forces are given by

$$\begin{aligned} L &= a_L(\dot{x}^2 + \dot{h}^2)(1 + c\alpha) \\ D &= a_D(\dot{x}^2 + \dot{h}^2)(1 + b(1 + c\alpha)^2) \end{aligned} \tag{19}$$

where a_L, a_D are dimensionless *lift* and *drag coefficients*, and b and c are positive constants. We assume that the autopilot has direct control over both the forward thrust T (throttle) and the aircraft pitch θ (through the elevators), thus there are two continuous control inputs $(u_1, u_2) = (T, \theta)$. Physical considerations impose constraints on the inputs:

$$u \in [T_{min}, T_{max}] \times [\theta_{min}, \theta_{max}] \tag{20}$$

The longitudinal dynamics may be modeled by the Newton-Euler equations:

$$M \begin{bmatrix} \ddot{x} \\ \ddot{h} \end{bmatrix} = R(\theta) \left[R^T(\alpha) \begin{bmatrix} -D \\ L \end{bmatrix} + \begin{bmatrix} T \\ 0 \end{bmatrix} \right] + \begin{bmatrix} 0 \\ -Mg \end{bmatrix} \tag{21}$$

where $R(\alpha)$ and $R(\theta)$ are standard rotation matrices, M is the mass of the aircraft, and g is gravitational acceleration. The state of the system is $\mathbf{x} = (x, \dot{x}, h, \dot{h})^T$.

The speed of the aircraft is defined as $V = \sqrt{\dot{x}^2 + \dot{h}^2}$. The simplified FMS studied here uses control inputs T and θ to control combinations of the speed V, flight path angle γ, and altitude h. The linear and angular accelerations $(\dot{V}, V\dot{\gamma})$ may be derived directly from (21):

$$\begin{aligned} \dot{V} &= -\frac{D}{M} - g\sin\gamma + \frac{T}{M}\cos\alpha \end{aligned} \tag{22}$$

$$\begin{aligned} V\dot{\gamma} &= \frac{L}{M} - g\cos\gamma + \frac{T}{M}\sin\alpha \end{aligned} \tag{23}$$
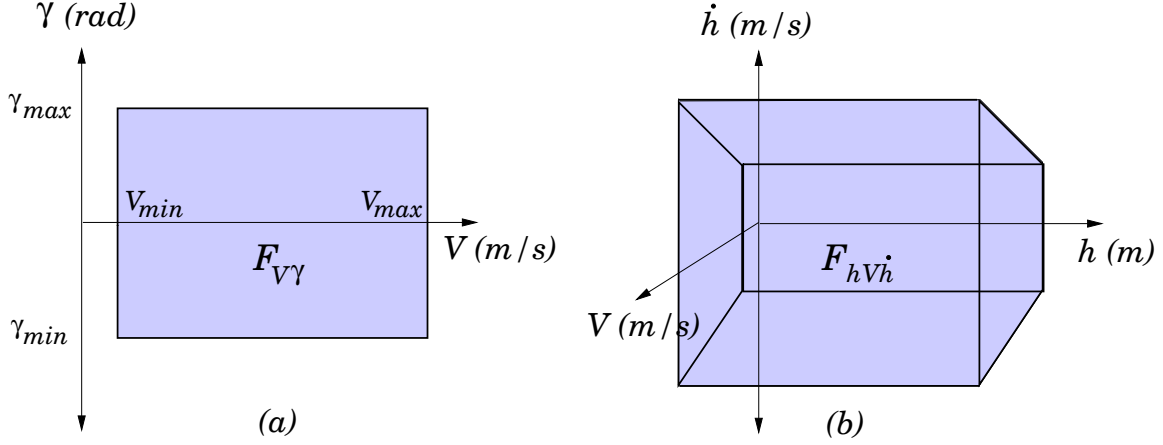
Figure 4: (a) Simplified Aerodynamic Flight Envelope in (V, γ)-space: axes are speed V, flight path angle γ; (b) Simplified Aerodynamic Flight Envelope in $(h, V, \dot{h})$-space: axes are altitude h, speed V, vertical speed $\dot{h}$.

Note that these dynamics are expressed solely in terms of (V, γ) and inputs (T, θ), where $\alpha = \theta - \gamma$; thus equations (22), (23) are a convenient way to represent the dynamics for modes in which h is not a controlled variable.

Safety regulations for the aircraft dictate that V, γ, and h must remain within specified limits:

$$
\begin{aligned}
V_{min} &\leq V \leq V_{max} \\
\gamma_{min} &\leq \gamma \leq \gamma_{max} \\
h_{min} &\leq h \leq h_{max}
\end{aligned}
\tag{24}
$$

where $V_{min}, V_{max}, \gamma_{min}, \gamma_{max}, h_{min}, h_{max}$ are functions of such factors as airspace regulations, type of aircraft, and weather. For aircraft flying in en-route airspace, we assume that these limits are constants, and thus the aerodynamic flight envelope F is as illustrated in Figure 4, in (V, γ)-space and $(h, V, \dot{h})$-space, where $\dot{h} = V \sin \gamma$. The state trajectory must remain within F at all times during en-route flight. We also impose a *secondary criterion*, that the state trajectory must satisfy constraints on the linear and angular acceleration:

$$
|\dot{V}| \leq 0.1g, \quad |V\dot{\gamma}| \leq 0.1g
\tag{25}
$$

imposed for passenger comfort.

In our calculations we use the following parameter values, which correspond to a DC-8 at cruising speed: $M = 85000$kg, $b = 0.01$, $c = 6$, $a_L = 30$, $a_D = 2$, $T_{min} = 40000$ N, $T_{max} = 80000$ N, $\theta_{min} = -22.5°$, $\theta_{max} = 22.5°$, $V_{min} = 180$ m/s, $V_{max} = 240$ m/s, $\gamma_{min} = -22.5°$ and $\gamma_{max} = 22.5°$. The bounds on the pitch angle θ and the flight path angle γ are chosen to be symmetric about zero for ease of computation. In actual flight systems, the positive bound on these angles is greater than the negative bound. Also, the angles chosen for this example are greater than what are considered acceptable for passenger flight ($\pm 10°$). Since we are interested in en route flight, the limits on the altitudes are: $h_{min} = 15,000$ feet, $h_{max} = 51,000$ feet.

In this lecture, we consider the specification $\square F_{V\gamma}$: the airspeed V and flight path angle γ must remain in the envelope $F_{V\gamma}$ at all times. We derive the maximal controlled invariant set contained

in $F_{V\gamma}$, using the (V, γ)-dynamics (22), (23):

$$\dot{V} = -\frac{D}{M} - g\sin\gamma + \frac{T}{M}\cos\alpha$$

$$V\dot{\gamma} = \frac{L}{M} - g\cos\gamma + \frac{T}{M}\sin\alpha$$

where $\alpha = \theta - \gamma$. Let

$$F_{V\gamma} = \{(V, \gamma) \mid \forall i \in \{1, 2, 3, 4\},\ l_i(V, \gamma) \geq 0\} \tag{26}$$

where

$$l_1(V, \gamma) = V - V_{min} \tag{27}$$
$$l_2(V, \gamma) = -\gamma + \gamma_{max} \tag{28}$$
$$l_3(V, \gamma) = -V + V_{max} \tag{29}$$
$$l_4(V, \gamma) = \gamma - \gamma_{min} \tag{30}$$

$\partial F_{V\gamma}$ is only piecewise smooth, contradicting the assumption of existence of a differentiable function $l : (V, \gamma) \to \mathbb{R}$ such that $\partial F_{V\gamma} = \{(V, \gamma) \mid l(V, \gamma) = 0\}$. We show that, for this example, the calculation can in fact be performed one edge of the boundary at a time: we derive a Hamilton-Jacobi equation for each l_i, and prove that the intersection of the resulting sets is the maximal controlled invariant subset of $F_{V\gamma}$. The subscript i in each J_i, H_i will indicate that the calculation is for boundary l_i.

Starting with $l_1(V, \gamma)$, consider the system (22), (23) over the time interval $[t, 0]$, where $t < 0$, with cost function

$$J_1((V, \gamma), u(\cdot), t) : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_- \to \mathbb{R} \tag{31}$$

such that $J_1((V, \gamma), u(\cdot), t) = l_1(V(0), \gamma(0))$. Since there are no disturbances in our model, the dynamic game of Lecture 12 reduces to an optimal control problem. The optimal cost is found by maximizing with respect to u:

$$J_1^*((V, \gamma), t) = \max_{u(\cdot)\in\mathcal{U}} J_1((V, \gamma), u(\cdot), t) \tag{32}$$

We seek to compute $W_1^* = \{(V, \gamma) \mid J_1^*(V, \gamma) \geq 0\}$, which are those (V, γ) for which there exists a control input which keeps the system to the right of $l_1(V, \gamma) = 0$. The optimal Hamiltonian is given by the following, where we have substituted into the dynamics the expressions for the lift L and drag D forces (19) (neglecting the quadratic term in D):

$$H_1^*((V, \gamma), p) = \max_{u\in U}[p_1(-\frac{a_D V^2}{M} - g\sin\gamma + \frac{1}{M}T) + p_2(\frac{a_L V(1 - c\gamma)}{M} - \frac{g\cos\gamma}{V} + \frac{a_L cV}{M}\theta)] \tag{33}$$

where $p = (p_1, p_2) \in \mathbb{R}^2$. The Hamilton-Jacobi equation describing the evolution of $J_1^*((V, \gamma), t)$ is obtained from the Hamilton-Jacobi equation of Lecture 13:

$$-\frac{\partial J_1^*(x, t)}{\partial t} = \min\{0, H_1^*((V, \gamma), \frac{\partial J_1^*((V, \gamma), t)}{\partial(V, \gamma)})\} \tag{34}$$

with boundary condition $J_1^*((V, \gamma), 0) = l_1((V, \gamma))$.

The optimal control at $t = 0$ is computed from equation (33). The optimal throttle input T may be calculated directly from this equation: $u_1^*(0) = T_{max}$ (since $p_1 > 0$ for the inward pointing normal).
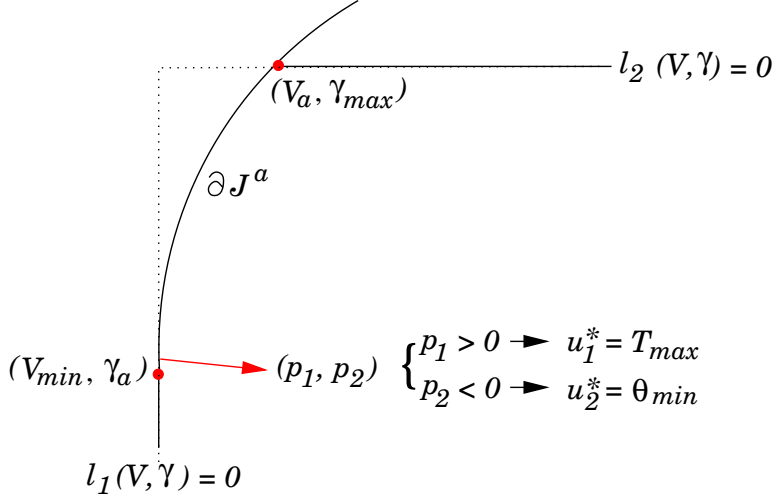
10

Figure 5: Computing the boundary ∂J^a.

The optimal pitch input must be calculated indirectly[1]. Define $(V_{min}, \gamma_a) = \{(V, \gamma) \mid l_1(V, \gamma) = 0 \cap H_1^*(V, \gamma) = 0\}$. Then:

$$\gamma_a = \sin^{-1}\left(\frac{T_{max}}{Mg} - \frac{a_D V_{min}^2}{Mg}\right) \tag{35}$$

Integrate the system dynamics (22), (23) with $(V(0), \gamma(0)) = (V_{min}, \gamma_a)$, $u = (u_1^*, u_2^*)$, backwards from $t = 0$ to $t = -T$, where T is chosen to be large enough so that the solution intersects $\{(V, \gamma) \mid l_2(V, \gamma) = 0\}$. The optimal control u_2^* is required for this calculation. At the abnormal extremal (V_{min}, γ_a), any $u_2 \in [\theta_{min}, \theta_{max}]$ may be used. However, as we integrate the system, we leave the abnormal extremal regardless of the choice of u_2 instantaneously, and u_2^* is uniquely determined. For all $u_2 \in [\theta_{min}, \theta_{max}]$, the inward pointing normal to the solution $(V(t), \gamma(t))$ of the system (22), (23), starting at (V_{min}, γ_a) and proceeding backwards in time for small $t < 0$ using $u_1 = u_1^*$, is such that p_2 is negative. Thus, $u_2^* = \theta_{min}$. Denote the point of intersection of the solution of (22), (23) with $\{(V, \gamma) \mid l_2(V, \gamma) = 0\}$ as (V_a, γ_{max}), and the solution to (22), (23) between (V_{min}, γ_a) and (V_a, γ_{max}) as ∂J^a, as shown in Figure 5. Repeat this calculation for the remaining three boundaries. Of the remaining three, only $\{(V, \gamma) \mid l_3(V, \gamma) = 0\}$ contains a point at which the associated optimal Hamiltonian, $H_3^*((V, \gamma), p)$, becomes zero. We denote this point as (V_{max}, γ_b) where:

$$\gamma_b = \sin^{-1}\left(\frac{T_{min}}{Mg} - \frac{a_D V_{max}^2}{Mg}\right) \tag{36}$$

and similarly calculate ∂J^b and V_b, as shown in Figure 7.

Lemma 3 *For the aircraft dynamics (22), (23) with flight envelope $F_{V\gamma}$ given by (26), and input constraints (20), the maximal controlled invariant subset of $F_{V\gamma}$, denoted $W_{V\gamma}^*$, is the set enclosed*

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[1]Since $H_1^*((V, \gamma), p)$ loses dependence on u_2 on the set $\{(V, \gamma) \mid l_1(V, \gamma) = 0\}$, the calculations involve computing the so-called *abnormal extremals* [9].
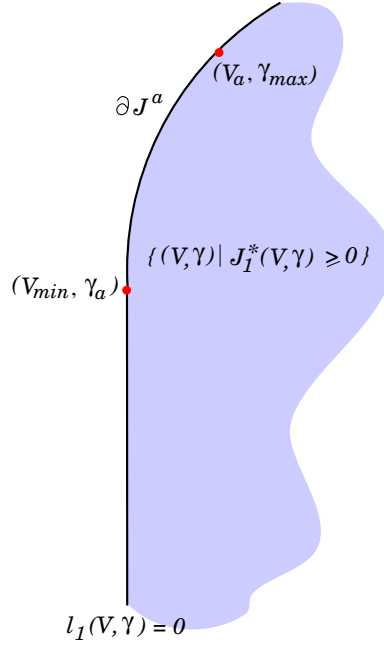
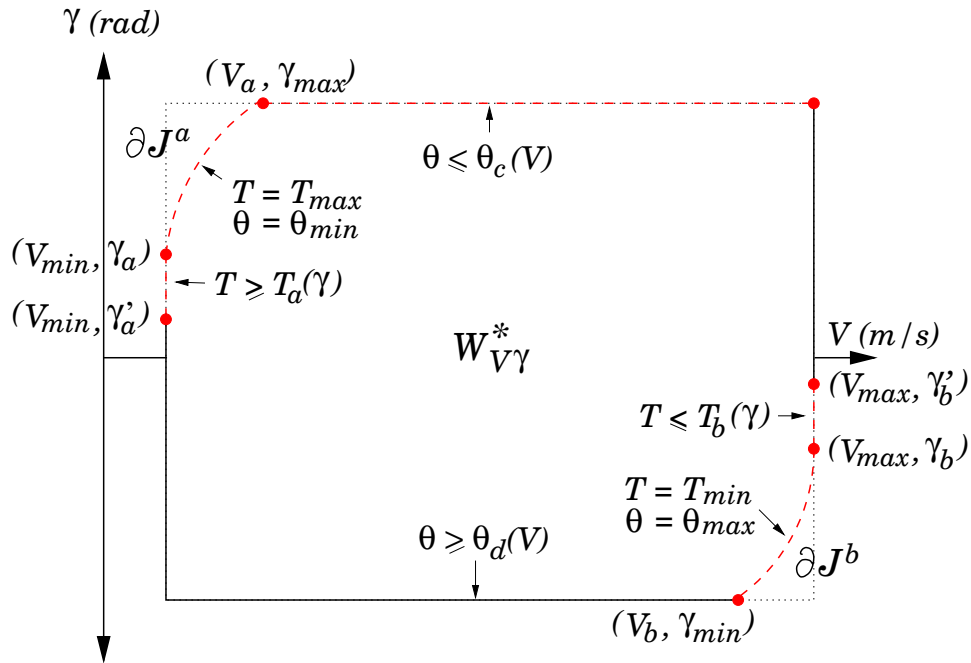Figure 6: Computing the set $\{(V, \gamma) \mid J_1^*(V, \gamma) = 0\}$.

Figure 7: The set $W_{V\gamma}^*$ in (V, γ)-space, with control law as indicated. Values used are for a DC-8: $\gamma_{min} = -\pi/8$ rad, $\gamma_{max} = \pi/8$ rad, $V_{min} = 180$ m/s, $V_{max} = 240$ m/s, $\theta_{min} = -\pi/8$ rad, $\theta_{max} = \pi/8$ rad, $T_{min} = 40$ kN, $T_{max} = 80$ kN.

by

$$\partial W^*_{V\gamma} = \{(V,\gamma) \mid \quad (V = V_{min}) \wedge (\gamma_{min} \le \gamma \le \gamma_a) \quad \vee$$
$$(V,\gamma) \in \partial J^a \qquad\qquad\qquad \vee$$
$$(\gamma = \gamma_{max}) \wedge (V_a \le V \le V_{max}) \quad \vee$$
$$(V = V_{max}) \wedge (\gamma_b \le \gamma \le \gamma_{max}) \quad \vee \tag{37}$$
$$(V,\gamma) \in \partial J^b \qquad\qquad\qquad \vee$$
$$(\gamma = \gamma_{min}) \wedge (V_{min} \le V \le V_b)\}$$

Proof: We first prove that the boundary of the set $\cap_{i \in \{1,2,3,4\}}\{(V,\gamma) \mid J^*_i(V,\gamma) \ge 0\}$ is the boundary constructed in equation (37). We then prove that this set is equal to $W^*_{V\gamma}$, the maximal controlled invariant set contained in $F_{V\gamma}$.

Consider first the edge $\{(V,\gamma) \mid l_1(V,\gamma) = 0\}$ in ∂F. We will show that

$$\{(V,\gamma) \mid J^*_1(V,\gamma) = 0\} = \{(V,\gamma) \mid (V = V_{min}) \wedge (\gamma_{min} \le \gamma \le \gamma_a)\} \cup \{(V,\gamma) \in \partial J^a\} \tag{38}$$

The optimal Hamiltonian $H^*_1((V,\gamma),p)$ satisfies:

$$H^*_1((V,\gamma),p) \begin{cases} < 0 & (V,\gamma) \in F_{V\gamma} \cap l_1(V,\gamma) = 0 \cap \gamma > \gamma_a \\ = 0 & (V,\gamma) \in F_{V\gamma} \cap l_1(V,\gamma) = 0 \cap \gamma = \gamma_a \\ > 0 & (V,\gamma) \in F_{V\gamma} \cap l_1(V,\gamma) = 0 \cap \gamma < \gamma_a \end{cases} \tag{39}$$

Thus, the set $\{(V,\gamma) \mid (V = V_{min}) \wedge (\gamma_{min} \le \gamma \le \gamma_a)\}$ remains unchanged under the evolution of the Hamilton-Jacobi equation (34), since $H^*_1 > 0$ for this set. We now prove that for $(V,\gamma) \in \partial J^a$, $J^*_1(V,\gamma) = 0$. $J^*_1(V,\gamma)$ satisfies:

$$\begin{bmatrix} \frac{\partial J^*_1(V,\gamma)}{\partial V} \\ \frac{\partial J^*_1(V,\gamma)}{\partial \gamma} \end{bmatrix} [-\frac{a_D V^2}{M} - g\sin\gamma + \frac{1}{M}T_{max}, \frac{a_L V(1-c\gamma)}{M} - \frac{g\cos\gamma}{V} + \frac{a_L cV}{M}\theta_{min}] = 0 \tag{40}$$

Since

$$\begin{bmatrix} \frac{\partial J^*_1(V,\gamma)}{\partial V} \\ \frac{\partial J^*_1(V,\gamma)}{\partial \gamma} \end{bmatrix} \tag{41}$$

is the inward pointing normal to $\{(V,\gamma) \mid J^*_1(V,\gamma) = 0\}$, then for each (V,γ) in $\{(V,\gamma) \mid J^*_1(V,\gamma) = 0\}$, the vector field

$$\begin{bmatrix} -\frac{a_D V^2}{M} - g\sin\gamma + \frac{1}{M}T_{max} \\ \frac{a_L V(1-c\gamma)}{M} - \frac{g\cos\gamma}{V} + \frac{a_L cV}{M}\theta_{min} \end{bmatrix} \tag{42}$$

is tangent to $\{(V,\gamma) \mid J^*_1(V,\gamma) = 0\}$. Thus the solution $(V(t),\gamma(t))$ to equations (22), (23) with $u = (T_{max}, \theta_{min})$ evolves along $J^*_1(V,\gamma) = 0$. Since, by construction, $(V,\gamma) \in \partial J^a$ satisfies equations (22), (23) with $u = (T_{max}, \theta_{min})$, then $(V,\gamma) \in \partial J^a$ satisfies $J^*_1(V,\gamma) = 0$.

Repeating this analysis for $\{(V,\gamma) \mid l_3(V,\gamma) = 0\}$, we can show that

$$\{(V,\gamma) \mid J^*_3(V,\gamma) = 0\} = \{(V,\gamma) \mid (V = V_{max}) \wedge (\gamma_b \le \gamma \le \gamma_{max})\} \cup \{(V,\gamma) \in \partial J^b\} \tag{43}$$

On the remaining boundaries, $H^*_2((V,\gamma),p) > 0$ and $H^*_4((V,\gamma),p) > 0$, so these boundaries remain unchanged under the evolution of their respective Hamilton-Jacobi equations.

It remains to prove that $W^*_{V\gamma} = \cap_{i \in \{1,2,3,4\}}\{(V,\gamma) \mid J^*_i(V,\gamma) \ge 0\}$. Clearly, any state (V,γ) for which there exists an i such that $J^*_i(V,\gamma) < 0$ must be excluded from $W^*_{V\gamma}$, since a trajectory

exists which starts from this state and drives the system out of $\cap_{i \in \{1,2,3,4\}}\{(V,\gamma) \mid J_i^*(V,\gamma) \geq 0\}$. Thus $W_{V\gamma}^* \subset \cap_{i \in \{1,2,3,4\}}\{(V,\gamma) \mid J_i^*(V,\gamma) \geq 0\}$. To prove equality, we need only show that at the points of intersection of the four boundaries: $\{(V_a,\gamma_{max}), (V_{max},\gamma_{max}), (V_b,\gamma_{min}), (V_{min},\gamma_{min})\}$ there exists a control input in U which keeps the system state inside $\cap_{i \in \{1,2,3,4\}}\{(V,\gamma) \mid J_i^*(V,\gamma) \geq 0\}$. Consider the point (V_a,γ_{max}). At this point, the set of control inputs which keeps the system state inside the set $\{(V,\gamma) \mid J_1^*(V,\gamma) \geq 0\}$ is $\{(T_{max},\theta_{min})\}$, and the set of control inputs which keeps the system state inside $\{(V,\gamma) \mid J_2^*(V,\gamma) \geq 0\}$ is the set $\{(T,\theta)|T \in [T_{min},T_{max}], \theta \in [\theta_{min}, \frac{M}{a_L V_a c}(\frac{g\cos\gamma_{max}}{V_a} - \frac{a_L V_a(1-c\gamma_{min})}{M})]\}$. Since these two sets have non-empty intersection, the intersection point $(V_a,\gamma_{max}) \in W_{V\gamma}^*$. Similar analysis holds for the remaining three intersection points. Thus $W_{V\gamma}^* = \cap_{i \in \{1,2,3,4\}}\{(V,\gamma) \mid J_i^*(V,\gamma) \geq 0\}$. ∎

Lemma 4 *The least restrictive controller that renders $W_{V\gamma}^*$ controlled invariant is $g(V,\gamma) = U \cap \hat{g}(V,\gamma)$, where:*

$$\hat{g}(V,\gamma) = \{ \begin{array}{ll} \emptyset & \text{if } (V,\gamma) \in (W_{V\gamma}^*)^c \\ T \geq T_a(\gamma) & \text{if } (V = V_{min}) \wedge (\gamma_{min} \leq \gamma \leq \gamma_a) \\ \theta = \theta_{min} \wedge T = T_{max} & \text{if } (V,\gamma) \in \partial J^a \\ \theta \leq \theta_c(V) & \text{if } (\gamma = \gamma_{max}) \wedge (V_a \leq V \leq V_{max}) \\ T \leq T_b(\gamma) & \text{if } (V = V_{max}) \wedge (\gamma_b \leq \gamma \leq \gamma_{max}) \\ \theta = \theta_{max} \wedge T = T_{min} & \text{if } (V,\gamma) \in \partial J^b \\ \theta \geq \theta_d(V) & \text{if } (\gamma = \gamma_{min}) \wedge (V_{min} \leq V \leq V_b)\} \end{array} \quad (44)$$

with

$$T_a(\gamma) = a_D V_{min}^2 + Mg\sin\gamma \tag{45}$$

$$T_b(\gamma) = a_D V_{max}^2 + Mg\sin\gamma \tag{46}$$

$$\theta_c(V) = \frac{M}{a_L V c}\left(\frac{g\cos\gamma_{max}}{V} - \frac{a_L V(1-c\gamma_{max})}{M}\right) \tag{47}$$

$$\theta_d(V) = \frac{M}{a_L V c}\left(\frac{g\cos\gamma_{min}}{V} - \frac{a_L V(1-c\gamma_{min})}{M}\right) \tag{48}$$

Proof: Consider the set $\{(V,\gamma) \mid (V = V_{min}) \wedge (\gamma_{min} \leq \gamma \leq \gamma_a)\}$. For each (V,γ) in this set, denote by $(T_a(\gamma),\theta_a(\gamma))$ the values of (T,θ) for which the vector field $(\dot{V},\dot{\gamma})$ becomes tangent to this set. These are the (T,θ) for which $\dot{V} = 0$: setting $\dot{V} = 0$ leads to equation (45) for all $\theta_a(\gamma) \in [\theta_{min},\theta_{max}]$. Thus, $\{[T_a(\gamma),T_{max}] \times [\theta_{min},\theta_{max}]\} \subseteq U$ keeps the system either tangent to or to the right side of the boundary $\{(V,\gamma) \mid (V = V_{min}) \wedge (\gamma_{min} \leq \gamma \leq \gamma_a)\}$. At the point (V_{min},γ_a'), where $T_a(\gamma_a') = T_{min}$ the vector field cone $(\dot{V},\dot{\gamma})$ for $(T,\theta) \in U$ points completely inside $F_{V\gamma}$. At γ_a, the cone points completely outside $F_{V\gamma}$, and $T = T_{max}$ is the unique value of throttle which keeps the system trajectory $(V(t),\gamma(t))$ tangent to $F_{V\gamma}$. This is illustrated in Figure 8, which shows the upper left boundary of $F_{V\gamma}$, and the cone of controls at the point (V_{min},γ_a).

The calculation may be repeated for the set $\{(V,\gamma) \mid (V = V_{max}) \wedge (\gamma_b \leq \gamma \leq \gamma_{max})\}$. Here, denote by $(T_b(\gamma),\theta_b(\gamma))$ the values of (T,θ) for which the vector field $(\dot{V},\dot{\gamma})$ becomes tangent to this set. Setting $\dot{V} = 0$ leads to equation (46) for all $\theta_b(\gamma) \in [\theta_{min},\theta_{max}]$. Therefore, $\{[T_{min},T_b(\gamma)] \times [\theta_{min},\theta_{max}]\} \subseteq U$ keeps the system either tangent to or to the left side of the boundary $\{(V,\gamma) \mid (V = V_{max}) \wedge (\gamma_b \leq \gamma \leq \gamma_{max})\}$. At the point (V_{max},γ_b), where $T_b(\gamma_b) = T_{min}$, T_{min} is the unique thrust which keeps the system trajectory tangent to $F_{V\gamma}$ (lower right boundary of $F_{V\gamma}$ in Figure 8).
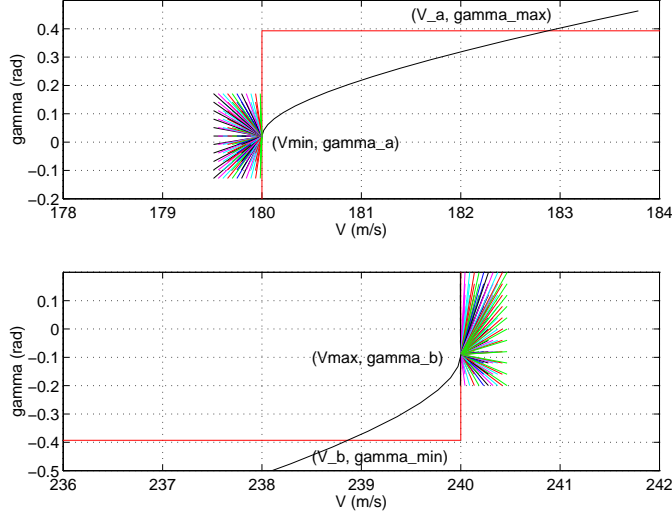
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Figure 8: Upper left boundary and lower right boundary of $F_{V\gamma}$.

Similar calculations along the upper and lower sides of $\partial F_{V\gamma}$ yield that the values of θ for which the vector field becomes tangent to $\partial F_{V\gamma}$ are $\theta_c(V)$ and $\theta_d(V)$ of equations (47) and (48). ∎

In Figure 7, the portions of $W^*_{V\gamma}$ for which all control inputs are safe ($g(V, \gamma) = U$) are indicated with solid lines; those for which only a subset are safe ($g(V, \gamma) \subset U$) are indicated with dashed lines. The map defines the *least restrictive safe control scheme* and determines the mode switching logic. On ∂J^a and ∂J^b, the system must be in **Mode 2** or **Mode 3**. Anywhere else in $W^*_{V\gamma}$, any of the three modes is valid as long as the input constraints of equation (44) are satisfied. In the regions $F_{V\gamma} \backslash W^*_{V\gamma}$ (the upper left and lower right corners of $F_{V\gamma}$), no control inputs will keep the system inside of $F_{V\gamma}$.

Additional Constraints for Passenger Comfort

Cost functions involving the linear and angular accelerations can be used to encode the requirement for passenger comfort (we use J_5, J_6 in the following, after J_1 to J_4 of the previous section):

$$J_5((V, \gamma), u(\cdot), t)) = -\max_{t \geq 0} |\dot{V}(t)|, \quad J_6((V, \gamma), u(\cdot), t)) = -\max_{t \geq 0} |V(t)\dot{\gamma}(t)| \tag{49}$$

The requirement that the linear and angular accelerations remain within the limits determined for comfortable travel are encoded by thresholds:

$$J_5((V, \gamma), u(\cdot), t)) \geq -0.1g, \quad J_6((V, \gamma), u(\cdot), t)) \geq -0.1g \tag{50}$$

Within the class of safe controls, a control scheme which addresses the passenger comfort requirement can be constructed. To do this, we solve the optimal control problem:

$$J_5^*((V, \gamma)) = \max_{u(\cdot) \in g(V, \gamma)} J_5, \quad J_6^*((V, \gamma)) = \max_{u(\cdot) \in g(V, \gamma)} J_6 \tag{51}$$

From this calculation, it is straightforward to determine the set of "comfortable" states:

$$\{(V, \gamma) \in W^*_{V\gamma} \mid J_5^*(V, \gamma) \geq -0.1g \wedge J_6^*(V, \gamma) \geq -0.1g\} \tag{52}$$

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The set of comfortable controls may be calculated by substituting the bounds on the accelerations into equation (22), (23) to get

$$
\begin{array}{ccccc}
-0.1Mg + a_D V^2 + Mg\sin\gamma & \leq & T & \leq & 0.1Mg + a_D V^2 + Mg\sin\gamma \\
-\frac{0.1Mg}{a_L V^2 c} - \frac{1-c\gamma}{c} + \frac{Mg\cos\gamma}{a_L V^2 c} & \leq & \theta & \leq & \frac{0.1Mg}{a_L V^2 c} - \frac{1-c\gamma}{c} + \frac{Mg\cos\gamma}{a_L V^2 c}
\end{array}
\tag{53}
$$

These constraints provide lower and upper bounds on the thrust and the pitch angle which may be applied at any point (V, γ) in $W_{V\gamma}^*$ while maintaining comfort.

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