EE40
Lecture 13
Prof. Chang-Hasnain

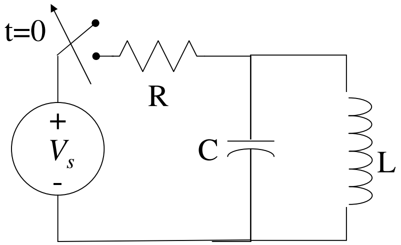
9/28/07

Reading: Chap. 5; appendix on complex numbers

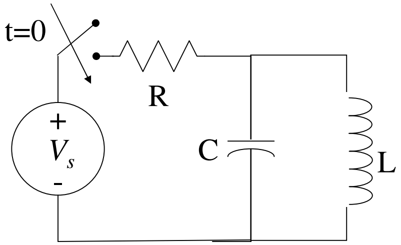
Chapter 5

- OUTLINE
  - – Phasors as notation for Sinusoids
  - – Arithmetic with Complex Numbers
  - – Complex impedances
  - – Circuit analysis using complex impdenaces
  - – Dervative/Integration as multiplication/division
  - – Phasor Relationship for Circuit Elements
- Reading
  - – Chap 5
  - – Appendix A

Example 1: 2nd Order RLC Circuit



Example 2: 2nd Order RLC Circuit



Sinusoidal Sources Create Too Much Algebra

$$x_P(t)+\tau\frac{dx_P(t)}{dt}=F_A\sin(wt)+F_B\cos(wt)$$

Guess a solution

Two terms to be general
Dervatives
Addition

$$x_P(t)=A\sin(wt)+B\cos(wt)$$

$$(A\sin(wt)+B\cos(wt))+\tau\frac{d(A\sin(wt)+B\cos(wt))}{dt}=F_A\sin(wt)+F_B\cos(wt)$$

$$(A-\tau B-F_A)\sin(wt)+(B+\tau A-F_B)\cos(wt)=0$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

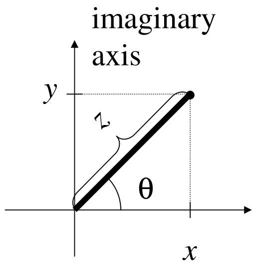
$$(A-\tau B-F_A)=0$$

$$(B+\tau A-F_B)=0$$

$$A=\frac{F_A+\tau F_B}{\tau^2+1}\qquad B=-\frac{\tau F_A-F_B}{\tau^2+1}$$

Phasors (vectors that rotate in the complex plane) are a clever alternative.

Complex Numbers (1)



$$j=\sqrt{(-1)}$$

- x is the real part
- y is the imaginary part
- z is the magnitude
- θ is the phase

$$x=z\cos\theta\qquad y=z\sin\theta$$

$$z=\sqrt{x^2+y^2}\qquad \theta=\tan^{-1}\frac{y}{x}$$

- Rectangular Coordinates
  Z = $x + jy$
- Polar Coordinates:
  Z = $z \angle \theta$
- Exponential Form:

$$\mathbf{Z}=\mathbf{Z}(\cos\theta+j\sin\theta)$$

$$1=1e^{j0}=1\angle 0°$$

$$\mathbf{Z}=|\mathbf{Z}|e^{j\theta}=ze^{j\theta}$$

$$j=1e^{j\frac{\pi}{2}}=1\angle 90°$$

Complex Numbers (2)

Euler's Identities

$$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{2j}$$

$$e^{j\theta}=\cos\theta+j\sin\theta$$

$$\left|e^{j\theta}\right|=\sqrt{\cos^2\theta+\sin^2\theta}=1$$

Exponential Form of a complex number

$$\mathbf{Z}=|\mathbf{Z}|e^{j\theta}=ze^{j\theta}=z\angle\theta$$

Arithmetic With Complex Numbers

- To compute phasor voltages and currents, we need to be able to perform computation with complex numbers.
  - Addition
  - Subtraction
  - Multiplication
  - Division
- (And later use multiplication by jω to replace
  - Diffrentiation
  - Integration

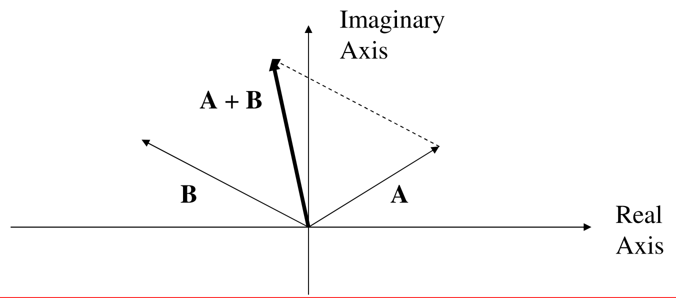
Addition

- Addition is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} + \mathbf{B} = (x + z) + j(y + w)$$



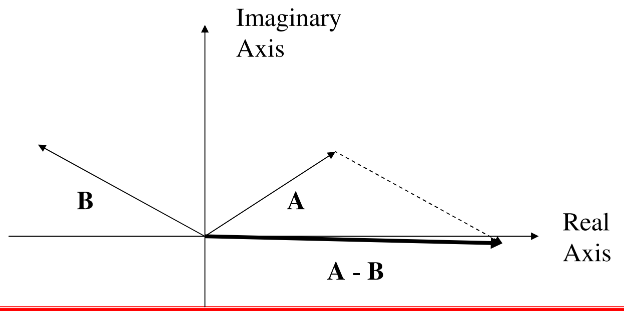
Subtraction

- Subtraction is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} - \mathbf{B} = (x - z) + j(y - w)$$



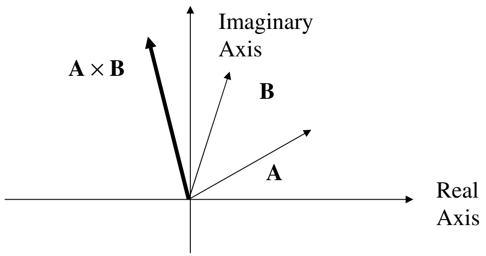
Multiplication

- Multiplication is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} \times \mathbf{B} = (A_M \times B_M) \angle (\theta + \phi)$$



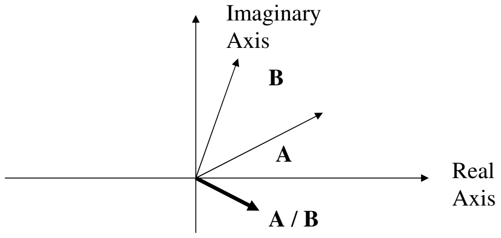
Division

- Division is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} / \mathbf{B} = (A_M / B_M) \angle (\theta - \phi)$$



Arithmetic Operations of Complex Numbers

- Add and Subtract: it is easiest to do this in rectangular format
  - Add/subtract the real and imaginary parts separately
- Multiply and Divide: it is easiest to do this in exponential/polar format
  - Multiply (divide) the magnitudes
  - Add (subtract) the phases

$$\mathbf{Z}_1 = z_1 e^{j\theta_1} = z_1\angle\theta_1 = z_1\cos\theta_1 + jz_1\sin\theta_1$$

$$\mathbf{Z}_2 = z_2 e^{j\theta_2} = z_2\angle\theta_2 = z_2\cos\theta_2 + jz_2\sin\theta_2$$

$$\mathbf{Z}_1 + \mathbf{Z}_2 = (z_1\cos\theta_1 + z_2\cos\theta_2) + j(z_1\sin\theta_1 + z_2\sin\theta_2)$$

$$\mathbf{Z}_1 - \mathbf{Z}_2 = (z_1\cos\theta_1 - z_2\cos\theta_2) + j(z_1\sin\theta_1 - z_2\sin\theta_2)$$

$$\mathbf{Z}_1 \times \mathbf{Z}_2 = (z_1 \times z_2) e^{j(\theta_1+\theta_2)} = (z_1 \times z_2)\angle(\theta_1+\theta_2)$$

$$\mathbf{Z}_1 / \mathbf{Z}_2 = (z_1 / z_2) e^{j(\theta_1-\theta_2)} = (z_1 / z_2)\angle(\theta_1-\theta_2)$$

Phasors

- Assuming a source voltage is a sinusoid time-varying function

$$v(t) = V\cos(\omega t + \theta)$$

- We can write:

$$v(t) = V\cos(\omega t + \theta) = V\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[Ve^{j(\omega t+\theta)}\right]$$

$$Define\ \ Phasor\ \ as\ \ Ve^{j\theta} = V\angle\theta$$

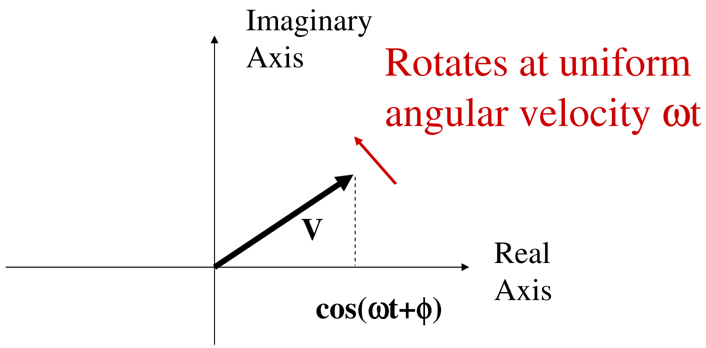
- Similarly, if the function is $v(t) = V\sin(\omega t + \theta)$

$$v(t) = V\sin(\omega t + \theta) = V\cos(\omega t + \theta - \frac{\pi}{2}) = \mathrm{Re}\left[Ve^{j(\omega t+\theta-\frac{\pi}{2})}\right]$$

$$Phasor\ = V\angle\left(\theta - \frac{\pi}{2}\right)$$

Phasor: Rotating Complex Vector

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{Ve^{j\phi}e^{jwt}\right\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right)$$

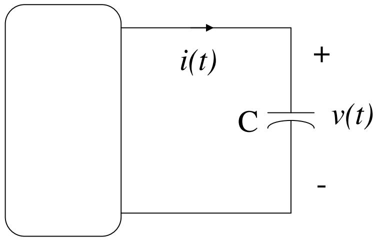


The head start angle is φ.

Complex Exponentials

- We represent a real-valued sinusoid as the real part of a complex exponential after multiplying by $e^{j\omega t}$.
- Complex exponentials
  - provide the link between time functions and phasors.
  - Allow derivatives and integrals to be replaced by multiplying or dividing by jω
  - make solving for AC steady state simple algebra with complex numbers.
- Phasors allow us to express current-voltage relationships for inductors and capacitors much like we express the current-voltage relationship for a resistor.

I-V Relationship for a Capacitor



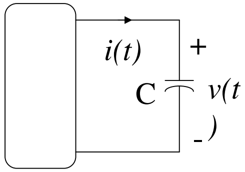
$$i(t) = C\frac{dv(t)}{dt}$$

Suppose that *v(t)* is a sinusoid:

$$v(t) = \mathrm{Re}\{V_M e^{j(\omega t+\theta)}\}$$

Find *i(t)*.

Capacitor Impedance (1)



$$i(t) = C\frac{dv(t)}{dt}$$

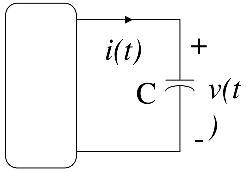
$$v(t) = V\boxed{\cos(\omega t+\theta)} = \frac{V}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right]$$

$$i(t) = C\frac{dv(t)}{dt} = \frac{CV}{2}\frac{d}{dt}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] = \frac{CV}{2}j\omega\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right]$$

$$= \frac{-\omega CV}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right] = -\omega CV\sin(\omega t+\theta) = \omega CV\boxed{\cos(\omega t+\theta+\frac{\pi}{2})}$$

$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\left(\theta+\frac{\pi}{2}\right)} = \frac{V}{\omega CV}\angle(\theta-\theta-\frac{\pi}{2}) = \frac{1}{\omega C}\angle(-\frac{\pi}{2}) = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

Capacitor Impedance (2)



$$i(t) = C\frac{dv(t)}{dt}$$

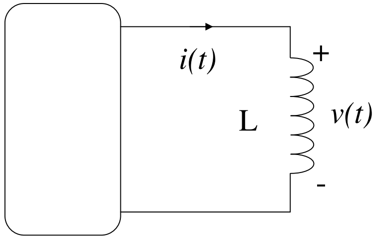
Phasor definition

$$v(t) = V\cos(\omega t+\theta) = \mathrm{Re}\left[Ve^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{V} = V\angle\theta$$

$$i(t) = C\frac{dv(t)}{dt} = \mathrm{Re}\left[CV\frac{de^{j(\omega t+\theta)}}{dt}\right] = \mathrm{Re}\left[j\omega CVe^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{I} = I\angle\theta$$

$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\theta} = \frac{V}{j\omega CV}\angle(\theta-\theta) = \frac{1}{j\omega C}$$

Inductor Impedance



$$v(t) = L\frac{di(t)}{dt}$$

$$\mathbf{V} = j\omega L\,\mathbf{I}$$

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Monday, 10/1/07

Reading: Chap. 5; appendix on complex numbers

Impedance

- AC steady-state analysis using phasors allows us to express the relationship between current and voltage using a formula that looks likes Ohm’s law:

$$\mathbf{V} = \mathbf{I}\,\mathbf{Z}$$

- **Z** is called impedance.

Computing the Current

Note: The differentiation and integration operations become algebraic operations

$$\frac{d}{dt} \Rightarrow j\omega \qquad \int dt \Rightarrow \frac{1}{j\omega}$$

Example

$$v(t) = 120\text{V} \cos(377t + 30°)$$

$$C = 2\mu\text{F}$$

- What is **V**?
- What is **I**?
- What is *i(t)*?

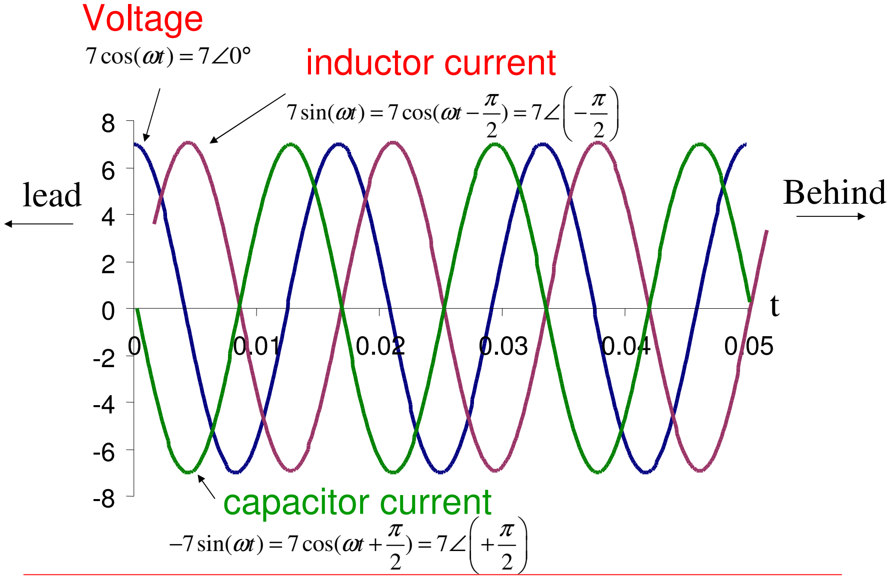
Example

$i(t) = 1\mu\text{A}\cos(2\pi\ 9.15\ 10^7 t + 30°)$

$L = 1\mu\text{H}$

- What is **I**?
- What is **V**?
- What is *v(t)*?

Phase



$7\cos(\omega t) = 7\angle 0°$

$7\sin(\omega t) = 7\cos(\omega t - \frac{\pi}{2}) = 7\angle\left(-\frac{\pi}{2}\right)$

$-7\sin(\omega t) = 7\cos(\omega t + \frac{\pi}{2}) = 7\angle\left(+\frac{\pi}{2}\right)$

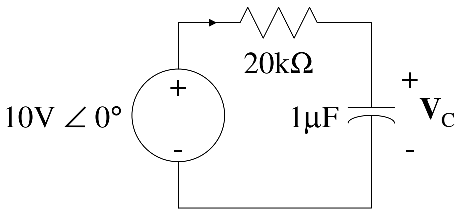
Phasor Diagrams

- A phasor diagram is just a graph of several phasors on the complex plane (using real and imaginary axes).
- A phasor diagram helps to visualize the relationships between currents and voltages.
- Capacitor: I leads V by 90°
- Inductor: V leads I by 90°

Some Thoughts on Impedance

- Impedance depends on the frequency ω.
- Impedance is (often) a complex number.
- Impedance allows us to use the same solution techniques for AC steady state as we use for DC steady state.

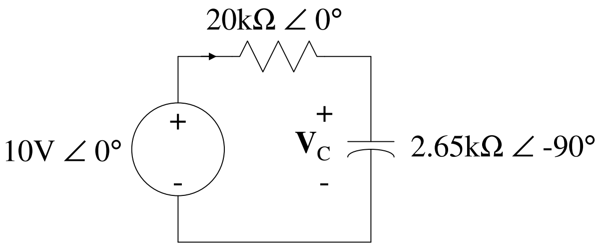
Example: Single Loop Circuit



f=60 Hz, $\mathbf{V}_C$=?

How do we find $\mathbf{V}_C$?
First compute impedances for resistor and capacitor:
$\mathbf{Z}_R$ = R= 20kΩ = 20kΩ ∠ 0°
$\mathbf{Z}_C$ = 1/*j* (2πf x 1μF) = 2.65kΩ ∠ -90°

Impedance Example

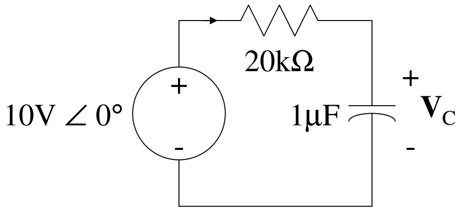


Now use the voltage divider to find $\mathbf{V}_C$:

$$\mathbf{V}_C = 10\mathrm{V}\angle 0°\left(\frac{2.65\mathrm{k}\Omega\angle -90°}{2.65\mathrm{k}\Omega\angle -90° + 20\mathrm{k}\Omega\angle 0°}\right)$$

$$\mathbf{V}_C = 1.31\mathrm{V}\angle -82.4°$$

What happens when ω changes?



ω = 10

Find $\mathbf{V}_C$

Circuit Analysis Using Complex Impedances

- Suitable for AC steady state.
- KVL

$$v_1(t)+v_2(t)+v_3(t)=0$$

$$V_1\cos(\omega t+\theta_1)+V_2\cos(\omega t+\theta_2)+V_3\cos(\omega t+\theta_3)=0$$

$$\mathrm{Re}\left[V_1e^{j(\omega t+\theta_1)}+V_2e^{j(\omega t+\theta_2)}+V_3e^{j(\omega t+\theta_3)}\right]=0$$

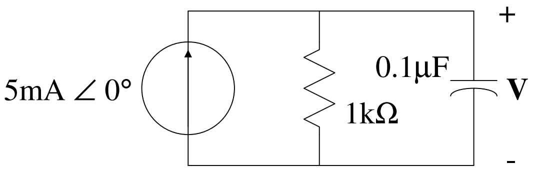
Phasor Form KVL

$$V_1e^{j(\theta_1)}+V_2e^{j(\theta_2)}+V_3e^{j(\theta_3)}=0$$

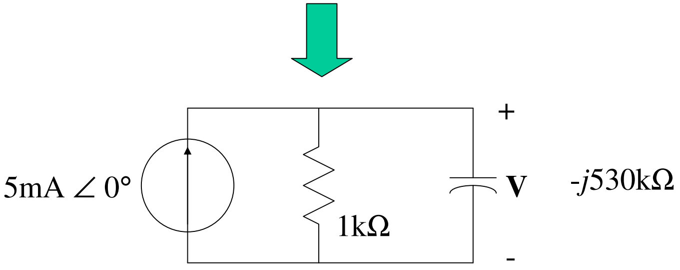
$$\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3=0$$

- Phasor Form KCL $\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3=0$
- Use complex impedances for inductors and capacitors and follow same analysis as in chap 2.

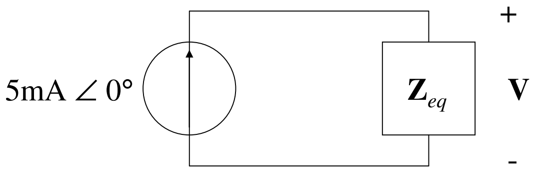
Steady-State AC Analysis



Find *v*(*t*) for ω=2π 3000



Find the Equivalent Impedance



$$\mathbf{Z}_{eq} = \frac{1000(-j530)}{1000 - j530} = \frac{10^3\angle 0^\circ \times 530\angle -90^\circ}{1132\angle -27.9^\circ}$$

$$\mathbf{Z}_{eq} = 468.2\Omega\angle -62.1^\circ$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = 5\text{mA}\angle 0^\circ \times 468.2\Omega\angle -62.1^\circ$$

$$\mathbf{V} = 2.34\text{V}\angle -62.1^\circ$$

$$v(t) = 2.34\text{V}\cos(2\pi 3000\text{t} - 62.1^\circ)$$

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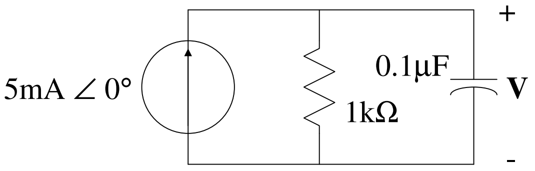
Prof. Chang-Hasnain

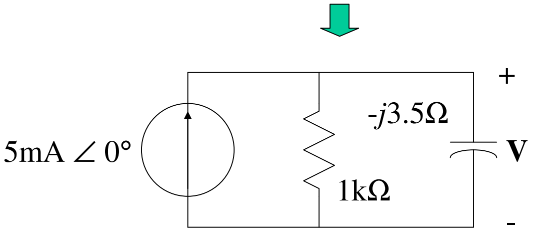
Wednesday, 10/3/07

Reading: Chap. 6; Supplementary Notes Ch. 1

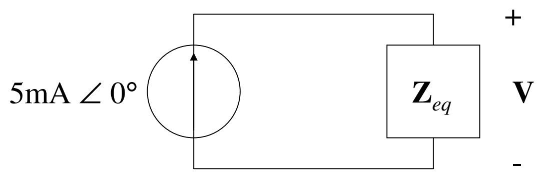
Change the Frequency



Find *v*(*t*) for ω=2π 455000



Find an Equivalent Impedance



$$\mathbf{Z}_{eq} = \frac{1000(-j3.5)}{1000 - j3.5} = \frac{10^3\angle 0^\circ \times 3.5\angle -90^\circ}{1000\angle -0.2^\circ}$$

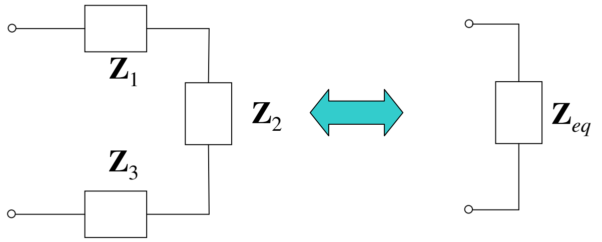
$$\mathbf{Z}_{eq} = 3.5\Omega\angle -89.8^\circ$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = 5\text{mA}\angle 0^\circ \times 3.5\Omega\angle -89.8^\circ$$

$$\mathbf{V} = 17.5\text{mV}\angle -89.8^\circ$$

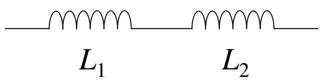
$$v(t) = 17.5\text{mV}\cos(2\pi 455000\text{t} - 89.8^\circ)$$

Series Impedance

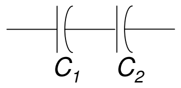


$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3$$

For example:

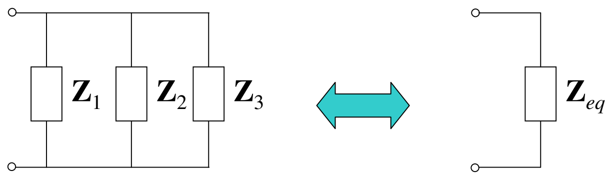


$$\mathbf{Z}_{eq} = j\omega(L_1 + L_2)$$



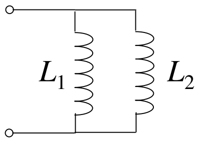
$$\mathbf{Z}_{eq} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}$$

Parallel Impedance

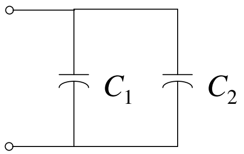


$$1/\mathbf{Z}_{eq} = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3$$

For example:

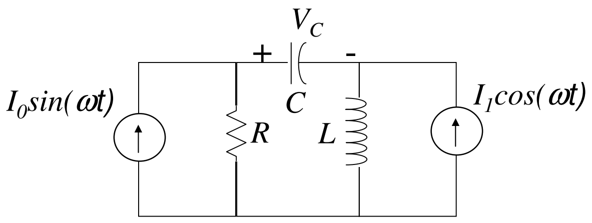


$$\mathbf{Z}_{eq} = j\omega\frac{L_1L_2}{(L_1 + L_2)}$$



$$\mathbf{Z}_{eq} = \frac{1}{j\omega(C_1 + C_2)}$$

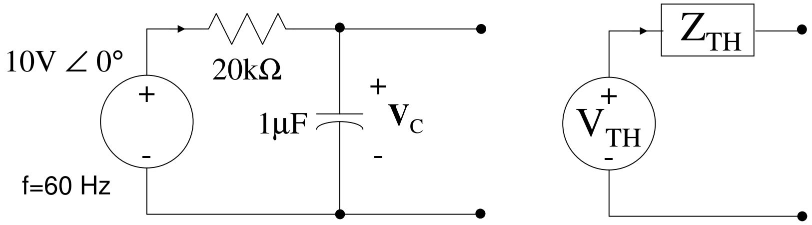
Steady-State AC Node-Voltage Analysis



- Try using Thevinin equivalent circuit.
- What happens if the sources are at different frequencies?

R	C	L
$v_0(t)=V_0\cos(\omega t)$	$v_0(t)=V_0\cos(\omega t)$	$v_0(t)=V_0\cos(\omega t)$
$\vec{V}_0=V_0\angle 0^\circ$	$\vec{V}_0=V_0\angle 0^\circ$	$\vec{V}_0=V_0\angle 0^\circ$
$i_0(t)=\frac{V_0}{R}\cos(\omega t)$	$i_0(t)=-\omega C V_0\sin(\omega t)$	$i_0(t)=\frac{V_0}{\omega L}\sin(\omega t)$
$\vec{I}_0=\frac{V_0}{R}\angle 0^\circ$	$\vec{I}_0=\omega C V_0\angle 90^\circ$	$\vec{I}_0=\frac{V_0}{\omega L}\angle -90^\circ$

Thevenin Equivalent



$$\mathbf{Z}_R = \text{R}= 20\text{k}\Omega = 20\text{k}\Omega \angle 0^\circ$$

$$\mathbf{Z}_C = 1/j\ (2\pi\text{f x } 1\mu\text{F}) = 2.65\text{k}\Omega \angle -90^\circ$$

$$\mathbf{V}_{TH} = \mathbf{V}_{OC} = 10\text{V} \angle 0^\circ \left(\frac{2.65\text{k}\Omega\angle -90^\circ}{2.65\text{k}\Omega\angle -90^\circ + 20\text{k}\Omega\angle 0^\circ} \right) = 1.31\angle -82.4$$

$$\mathbf{Z}_{TH} = \mathbf{Z}_R \parallel \mathbf{Z}_\text{C} = {}^\circ\left(\frac{20\text{k}\Omega\angle 0^\circ \cdot 2.65\text{k}\Omega\angle -90^\circ}{2.65\text{k}\Omega\angle -90^\circ + 20\text{k}\Omega\angle 0^\circ} \right) = 2.62\angle -82.4$$

Root Mean Square (rms) Values

- rms valued defined as

$$v_{RMS} = \sqrt{\frac{1}{T}\int_o^T v^2(t)dt} \qquad \text{T = period}$$

- Assuming a sinusoid gives

$$v_{RMS} = \sqrt{\frac{1}{T}\int_o^T v_m{}^2\cos^2(\omega t+\theta)dt}$$

- Using an identity gives

$$v_{RMS} = \sqrt{\frac{v_m{}^2}{2T}\int_o^T [1+\cos(2\omega t+2\theta)]dt}$$

- Evaluating at limits gives

$$v_{RMS} = \sqrt{\frac{v_m{}^2}{2T}[T+\frac{1}{2\omega}\sin(2\omega T+2\theta)-\frac{1}{2w}\sin(2\theta)]} \qquad v_{RMS} = \frac{v_m}{\sqrt{2}}$$

Power: Instantaneous and Time-Average

For a Resistor

- The instantaneous power is

$$p(t) = v(t)i(t) = \frac{v(t)^2}{R}$$

- The time-average power is

$$P_{AVE} = \frac{1}{T}\int_0^T \underset{v(t)i(t}{p(t)}dt = \frac{1}{T}\int_0^T \frac{v(t)^2}{R}dt = \frac{1}{R}[\frac{1}{T}\int_0^T v(t)^2 dt] = \frac{v_{rms}^2}{R}$$

For an Impedance

- The instantaneous power is

$$p(t) = v(t)i(t)$$

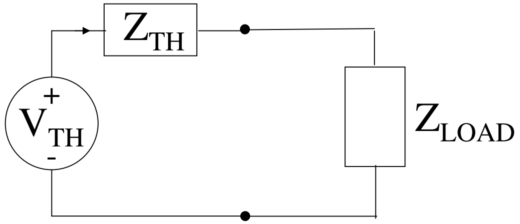
- The time-average power is

$$P_{AVE} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T v(t)i(t)dt = \text{Re}\{\mathbf{V}_{rms}\cdot\mathbf{I}_{rms}{}^*\}$$

- The reactive power at 2ω is

$$Q = \text{Im}\{\mathbf{V}_{rms}\cdot\mathbf{I}^*_{rms}\} \qquad P_{AVE}^2 + Q^2 = (V_{rms}\cdot I_{rms})^2$$

Maximum Average Power Transfer



- Maximum time average power occurs when

$$\mathbf{Z}_{LOAD} = \mathbf{Z}_{TH}^{*}$$

- This presents a resistive impedance to the source

$$\mathbf{Z}_{total} = \mathbf{Z}_{TH} + \mathbf{Z}_{TH}^{*}$$

- Power transferred is

$$P_{AVE} = \mathrm{Re}\{\mathbf{VI}^{*}\} = \mathrm{Re}\{\mathbf{V}\frac{\mathbf{V}^{*}}{2R}\} = \tfrac{1}{2}\frac{V_{rms}^{2}}{R}$$

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Lecture 16
Prof. Chang-Hasnain

Friday, 10/5/07

Reading: Chap. 6; Supplementary Notes Ch. 1

Chapter 6

- OUTLINE
  - Frequency Response for Characterization
  - Asymptotic Frequency Behavior
  - Log magnitude vs log frequency plot
  - Phase vs log frequency plot
  - dB scale
  - Transfer function example

Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of power levels, i.e. relative power levels.
  - The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunications pioneer.
  - The bel is a logarithmic measure. The number of bels for a given ratio of power levels is calculated by taking the logarithm, to the base 10, of the ratio.
  - one bel corresponds to a ratio of 10:1.
  - $B = \log_{10}(P_1/P_2)$ where P_1 and P_2 are power levels.
- The bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.
  - $1\mathrm{dB} = 10 \log_{10}(P_1/P_2)$
- dB are used to measure
  - Electric power, Gain or loss of amplifiers, Insertion loss of filters.

Logarithmic Measure for Power

- To express a power in terms of decibels, one starts by choosing a reference power, $P_{reference}$, and writing

  Power P in decibels = $10 \log_{10}(P/P_{reference})$
- Exercise:
  - Express a power of 50 mW in decibels relative to 1 watt.
  - $P\ (dB) = 10 \log_{10} (50 \times 10^{-3}) = -13$ dB
- Exercise:
  - Express a power of 50 mW in decibels relative to 1 mW.
  - $P\ (dB) = 10 \log_{10} (50) = 17$ dB.
- dBm to express absolute values of power relative to a milliwatt.
  - dBm = $10 \log_{10}$ (power in milliwatts / 1 milliwatt)
  - 100 mW = 20 dBm
  - 10 mW = 10 dBm

Logarithmic Measures for Voltage or Current

From the expression for power ratios in decibels, we can readily derive the corresponding expressions for voltage or current ratios.

Suppose that the voltage V (or current I) appears across (or flows in) a resistor whose resistance is R. The corresponding power dissipated, P, is V^2/R (or I^2R). We can similarly relate the reference voltage or current to the reference power, as

$$P_{reference} = (V_{reference})^2/R \text{ or } P_{reference} = (I_{reference})^2 R.$$

Hence,

$$\boxed{\begin{array}{c}\text{Voltage, V in decibels} = 20\log_{10}(V/V_{reference}) \\ \text{Current, I, in decibels} = 20\log_{10}(I/I_{reference})\end{array}}$$

Logarithmic Measures for Voltage or Current

Note that the voltage and current expressions are just like the power expression except that they have 20 as the multiplier instead of 10 because power is proportional to the square of the voltage or current.

Exercise: How many decibels larger is the voltage of a 9-volt transistor battery than that of a 1.5-volt AA battery? Let $V_{reference} = 1.5$. The ratio in decibels is

$$20 \log_{10}(9/1.5) = 20 \log_{10}(6) = 16 \text{ dB}.$$

Logarithmic Measures for Voltage or Current

The gain produced by an amplifier or the loss of a filter is often specified in decibels.

The input voltage (current, or power) is taken as the reference value of voltage (current, or power) in the decibel defining expression:

$$\boxed{\begin{array}{c}\text{Voltage gain in dB} = 20 \log_{10}(V_{output}/V_{input}) \\ \text{Current gain in dB} = 20\log_{10}(I_{output}/I_{input} \\ \text{Power gain in dB} = 10\log_{10}(P_{output}/P_{input})\end{array}}$$

Example: The voltage gain of an amplifier whose input is 0.2 mV and whose output is 0.5 V is

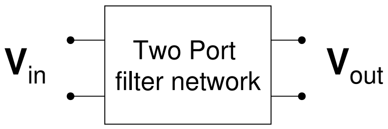
$$20\log_{10}(0.5/0.2\times10^{-3}) = 68 \text{ dB}.$$

Bode Plot

- Plot of magnitude of transfer function vs. frequency
  - Both x and y scale are in log scale
  - Y scale in dB
- Log Frequency Scale
  - Decade → Ratio of higher to lower frequency = 10
  - Octave → Ratio of higher to lower frequency = 2

Transfer Function

- Transfer function is a function of frequency
  - Complex quantity
  - Both magnitude and phase are function of frequency

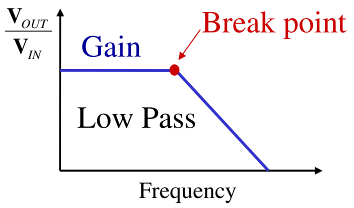


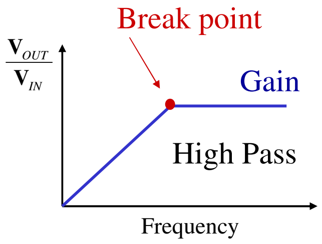
$$\mathbf{H}(f)=\frac{\mathbf{V_{out}}}{\mathbf{V_{in}}}=\frac{V_{out}}{V_{in}}\angle(\theta_{out}-\theta_{in})$$

$$\mathbf{H(f)}=H(f)\angle\theta$$

Frequency Response

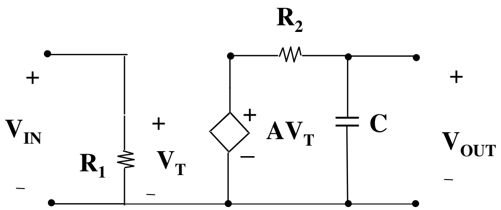
- The shape of the frequency response of the complex ratio of phasors $\mathbf{V}_{OUT}/\mathbf{V}_{IN}$ is a convenient means of classifying a circuit behavior and identifying key parameters.





FYI: These are log ratio vs log frequency plots

Example Circuit



$$TransferFunction=\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}}$$

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}}=\frac{AZ_c}{Z_R+Z_c}$$

A = 100

R_1 = 100,000 Ohms

R_2 = 1000 Ohms

C = 10 uF

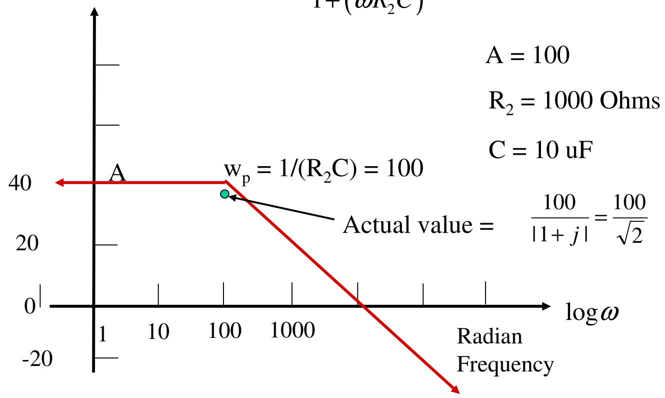
$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}}=\frac{A(1/jwC)}{R_2+1/j\omega C)}=\frac{A}{(1+j\omega R_2C)} \qquad H(\omega)=\frac{A}{1+j\omega R_2C}$$

First Order Bode Magnitude Plot

- The break frequency ω_B
  - is a property of the filter that can be found by examining the transfer function. It describes the frequency where the trends on the Bode plot are broken, where one trend (when) ends and the next (when) begins.
- We can break down the transfer function into 3 regimes, depending on the frequency:
  - $\omega = \omega_B$, $\omega >> \omega_B$, $\omega << \omega_B$
- At break frequency is also half-power frequency for first-order circuit
  - **$10\log_{10}(P_{\text{half-power}}/P_{\text{resonance}}) = 10\log_{10}(1/2) = -3$ dB.**

Bode Magnitude Plot – First-order Low-Pass Filter

$$y = 20\log|H(\omega)| = 20\log A + 10\log\frac{1}{1+(\omega R_2 C)^2}$$



A = 100

R_2 = 1000 Ohms

C = 10 uF

$w_p = 1/(R_2C) = 100$

Actual value = $\frac{100}{|1+j|} = \frac{100}{\sqrt{2}}$

What are the slopes of the Bode magnitude plot?

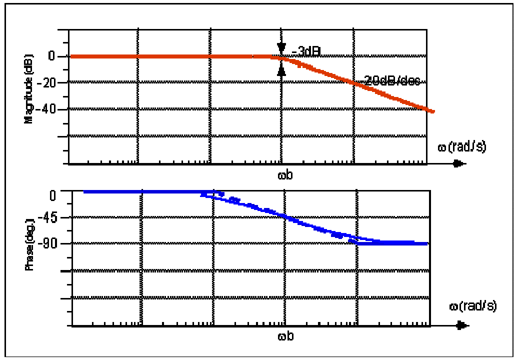
High-frequency asymptote of Low-pass filter

The high frequency asymptote of magnitude Bode plot assumes -20dB/decade slope

As $f \rightarrow \infty$

$$H(f) = \left(\frac{f}{f_B}\right)^{-1}$$

$$20\log_{10}\frac{H(10f_B)}{H(f_B)} = -20dB$$



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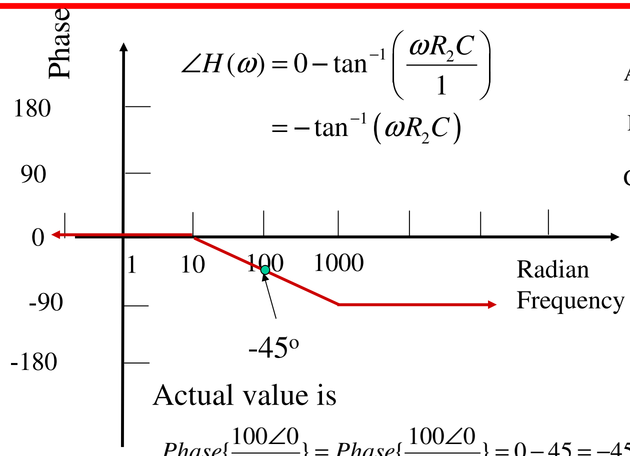
Reading: Chap. 6; Supplementary Notes Ch. 1

Bode Phase Plot – First-order Low-pass Filter



$$\angle H(\omega) = 0 - \tan^{-1}\left(\frac{\omega R_2 C}{1}\right)$$
$$= -\tan^{-1}\left(\omega R_2 C\right)$$

A = 100

R_2 = 1000 Ohms

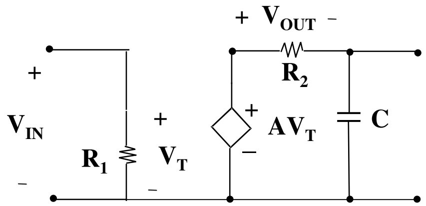
C = 10 uF

Actual value is

$$Phase\{\frac{100\angle 0}{|1+j|}\} = Phase\{\frac{100\angle 0}{\sqrt{2}\angle 45}\} = 0 - 45 = -45$$

What are the limiting values?

Example Circuit: 2



What is the transfer function if the output is taken across R_2? What filter should we get?

$$TransferFunction = \frac{V_{OUT}}{V_{IN}}$$

A = 100

R_1 = 100,000 Ohms

R_2 = 1000 Ohms

C = 10 uF

Filters

- Circuit designed to retain a certain frequency range and discard others
  - *Low-pass*: pass low frequencies and reject high frequencies
  - *High-pass*: pass high frequencies and reject low frequencies
  - *Band-pass*: pass some particular range of frequencies, reject other frequencies outside that band
  - *Notch*: reject a range of frequencies and pass all other frequencies

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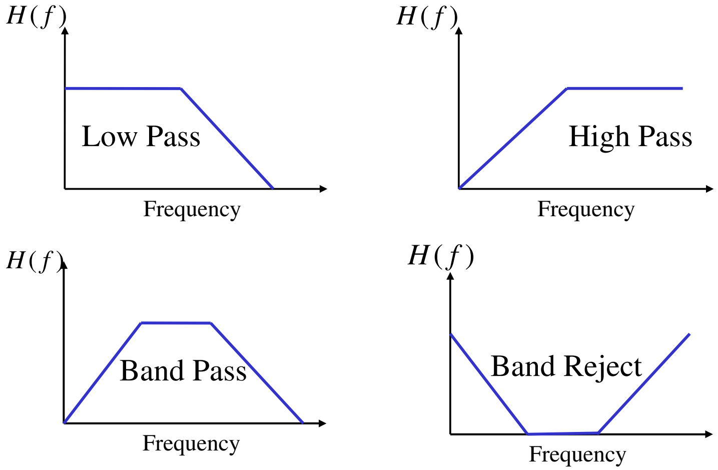
Common Filter Transfer Function vs. Freq



First-Order Lowpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V_C}}{\mathbf{V}} = \frac{1/(j\omega C)}{1/(j\omega C)+R} = \frac{1}{1+j\omega RC} = \frac{1}{\sqrt{1+(\omega RC)^2}}\angle -\tan^{-1}(\omega RC)$$

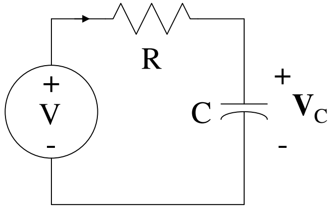
$$Let\ \omega_B = \frac{1}{RC} \quad and\ f_B = \frac{1}{2\pi RC}$$

$$\mathbf{H(f)} = H(f)\angle\theta$$

$$H(f) = \frac{1}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}}\ ,\theta = -\tan^{-1}\left(\frac{f}{f_B}\right)$$

$$H(f_B) = \frac{1}{\sqrt{2}} = 2^{-1/2}$$

$$20\log_{10}\frac{H(f_B)}{H(0)} = 20(-\frac{1}{2})\log_{10}2 = -3\ dB$$



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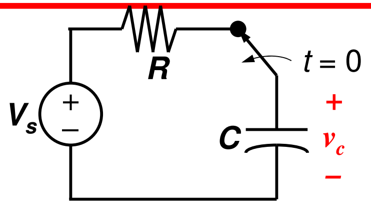
Reading: Chap. 6; Supplementary Notes Ch. 1

Recall Time-Domain Analysis



- Given $v_c(0-)=1$, *Vs=2 cos(wt), w*=200.
- *Find i(t),* $v_c(t)$=?

The Particular Solution: F(t) Sinusoid

$$x_P(t) + \tau\frac{dx_P(t)}{dt} = F_A\sin(wt) + F_B\cos(wt)$$

Guess a solution $x_P(t) = A\sin(wt) + B\cos(wt)$

$$(A\sin(wt)+B\cos(wt)) + \tau\frac{d(A\sin(wt)+B\cos(wt))}{dt} = F_A\sin(wt) + F_B\cos(wt)$$

$$(A-\tau\omega B-F_A)\sin(\omega t) + (B+\tau\omega A-F_B)\cos(\omega t) = 0$$

$$(A-\tau\omega B-F_A) = 0 \qquad (B+\tau\omega A-F_B) = 0$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$A = \frac{F_A+\tau\omega F_B}{(\tau\omega)^2+1} \qquad B = -\frac{\tau\omega F_A - F_B}{(\tau\omega)^2+1}$$

$$x_P(t) = \frac{F_A}{\sqrt{(\tau\omega)^2+1}}\left[\frac{\sin(\omega t)-\tau\omega\cos(\omega t)}{\sqrt{(\tau\omega)^2+1}}\right] + \frac{F_B}{\sqrt{(\tau\omega)^2+1}}\left[\frac{\tau\omega\sin(\omega t)+\cos(\omega t)}{\sqrt{(\tau\omega)^2+1}}\right]$$

$$= \frac{F_A\sin(\omega t-\theta)+F_B\cos(\omega t-\theta)}{\sqrt{(\tau\omega)^2+1}}; \quad where\ \theta = \tan^{-1}(\tau\omega)$$

The Particular Solution: F(t) Sinusoid -2

Assume $F_A=0$ $f(t)=F_B\cos\omega t,\; x_p(t)=\dfrac{F_B\cos(\omega t-\theta)}{\sqrt{(\tau\omega)^2+1}}$

in Phasor Form,

$$\vec{F}=F_B,\;\vec{X}_p=\frac{F_B}{\sqrt{(\tau\omega)^2+1}}\angle\theta$$

$$H(\omega)=\frac{1}{\sqrt{(\tau\omega)^2+1}}\angle\theta$$

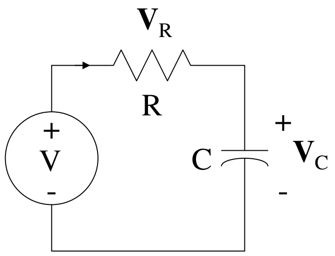
First-Order Highpass Filter

$$\mathbf{H(f)}=\frac{\mathbf{V}_R}{\mathbf{V}}=\frac{R}{1/(j\omega C)+R}=\frac{j\omega RC}{1+j\omega RC}=\frac{(\omega RC)}{\sqrt{1+(\omega RC)^2}}\angle\left[\frac{\pi}{2}-\tan^{-1}(\omega RC)\right]$$

$$H(f)=\frac{\left(\dfrac{f}{f_B}\right)}{\sqrt{1+\left(\dfrac{f}{f_B}\right)^2}}\;,\theta=\frac{\pi}{2}-\tan^{-1}\left(\frac{f}{f_B}\right)$$

$$H(f_B)=\frac{1}{\sqrt{2}}=2^{-1/2}$$

$$20\log_{10}\frac{H(f_B)}{H(0)}=20(-\frac{1}{2})\log_{10}2=-3\;dB$$



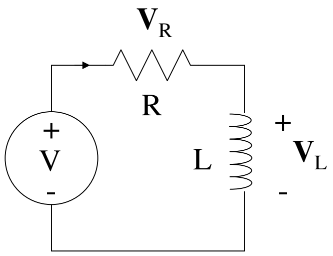
First-Order Lowpass Filter

$$\mathbf{H(f)}=\frac{\mathbf{V}_R}{\mathbf{V}}=\frac{1}{\dfrac{j\omega L}{R}+1}=\frac{1}{\sqrt{1+\left(\dfrac{\omega L}{R}\right)^2}}\angle-\tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$Let\;\omega_B=\frac{R}{L}\quad and\;\; f_B=\frac{R}{2\pi L}$$

$$\mathbf{H(f)}=H(f)\angle\theta$$

$$H(f)=\frac{1}{\sqrt{1+\left(\dfrac{f}{f_B}\right)^2}}\;,\theta=-\tan^{-1}\left(\frac{f}{f_B}\right)$$



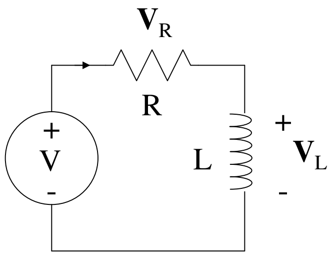
First-Order Highpass Filter

$$\mathbf{H(f)}=\frac{\mathbf{V}_L}{\mathbf{V}}=\frac{\dfrac{j\omega L}{R}}{\dfrac{j\omega L}{R}+1}=\frac{\dfrac{\omega L}{R}}{\sqrt{1+\left(\dfrac{\omega L}{R}\right)^2}}\angle\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{\omega L}{R}\right)\right]$$

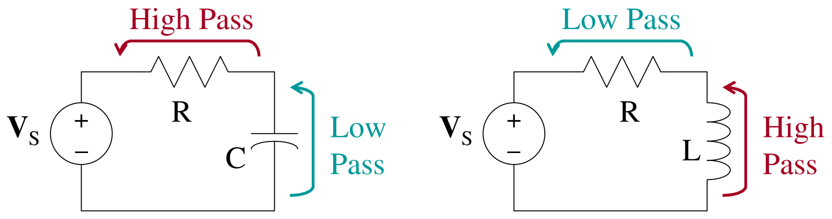
$$Let\;\omega_B=\frac{R}{L}\quad and\;\; f_B=\frac{R}{2\pi L}$$

$$\mathbf{H(f)}=H(f)\angle\theta$$

$$H(f)=\frac{\left(\dfrac{f}{f_B}\right)}{\sqrt{1+\left(\dfrac{f}{f_B}\right)^2}}\;,\theta=\frac{\pi}{2}-\tan^{-1}\left(\frac{f}{f_B}\right)$$



First-Order Filter Circuits



$\mathbf{H_R} = \mathrm{R} / (\mathrm{R} + 1/j\omega C)$ $\mathbf{H_R} = \mathrm{R} / (\mathrm{R} + j\omega \mathrm{L})$

$\mathbf{H_C} = (1/j\omega C) / (\mathrm{R} + 1/j\omega C)$ $\mathbf{H_L} = j\omega \mathrm{L} / (\mathrm{R} + j\omega \mathrm{L})$

Change of Voltage or Current with A Change of Frequency

One may wish to specify the change of a quantity such as the output voltage of a filter when the frequency changes by a factor of 2 (an octave) or 10 (a decade).

For example, a single-stage RC low-pass filter has at frequencies above ω = 1/RC an output that changes at the rate -20dB per decade.

Low-frequency asymptote of High-pass filter

$As\ f \to 0$

$$H(f) = \frac{\left(\dfrac{f}{f_B}\right)}{\underset{f\to\infty}{\sqrt{1+\left(\dfrac{f}{f_B}\right)^2}}} \to \left(\frac{f}{f_B}\right)$$

$$20\log_{10}\frac{H(f_B)}{H(0.1 f_B)} = 20dB$$

The low frequency asymptote of magnitude Bode plot assumes 20dB/decade slope

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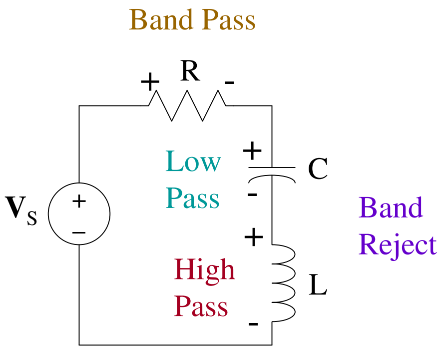
10/15/07

Reading: Chap. 6; Supplementary Notes Ch. 1

Second-Order Filter Circuits



$$\mathbf{Z} = R + 1/j\omega C + j\omega L$$

$$\mathbf{H}_R(\omega) = R / \mathbf{Z}$$

$$\mathbf{H}_C(\omega) = (1/j\omega C) / \mathbf{Z}$$

$$\mathbf{H}_L(\omega) = j\omega L / \mathbf{Z}$$

$$\mathbf{H}_{BR} + \mathbf{H}_{LP} + \mathbf{H}_{HP} = 1$$

Second-Order Low-Pass Filter

$$H_c(f) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C} + j\omega L} = \frac{\frac{1}{j\omega RC}}{1 + \frac{1}{j\omega RC} + j\omega\frac{L}{R}}$$

$$Let\ \omega_0^2 = \frac{1}{LC}\ ,\ Hence\ \omega_0 L = \frac{1}{\omega_0 C};\ define\ \frac{\omega_0 L}{R} = \frac{1}{R\omega_0 C} = Q_s$$

$$j\omega\frac{L}{R} = \frac{j\omega}{\omega_0}\frac{\omega_0 L}{R} = \frac{j\omega}{\omega_0}Q_s\ ,\ \frac{1}{j\omega RC} = \frac{\omega_0}{j\omega}\frac{1}{\omega_0 RC} = -\frac{j\omega_0}{\omega}Q_s$$

$$H_c(f) = \frac{-jQ_s\frac{\omega_0}{\omega}}{1 + jQ_s(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})}$$

$$\left|H_c(f)\right|^2 = \frac{\left(Q_s\frac{f_0}{f}\right)^2}{1 + \left[Q_s(\frac{f}{f_0} - \frac{f_0}{f})\right]^2}$$

Second-Order Low-Pass Filter

- $Low\ f : \frac{f_0}{f}$ dominates $H_c(f) = 1 \Rightarrow y = 10\log\left|H_c(f)\right|^2 = 0dB$

  $$\angle H_c(f) = 0°$$

- $High\ f : \frac{f}{f_0}$ dominates $H_c(f) \approx \frac{-jQ_s\frac{f_0}{f}}{jQ_s\frac{f}{f_0}} = -\left(\frac{f_0}{f}\right)^2,\ \left|H_c(f)\right|^2 \sim \left(\frac{f_0}{f}\right)^4$

  $$y = 10\log\left|H_c(f)\right|^2 \approx K - 40\log f \quad \Rightarrow Slope : -40dB/dec$$

  $$\angle H_c(f) = -180°$$

- $at\ f = f_0 :\ H_c(f) = -jQ_s\ ,\ \left|H_c(f)\right|^2 = Q_s^2 \Rightarrow y = 20\log Q_s$

  $$H_c(f) = -90°$$

Connection to the Time-Domain Analysis

- **Note: This resonant frequency ω_0 is the same formula we got using second-order differential equation (Surprise or not a surprise?)**
- **Also,** $2\alpha = \frac{R}{L}; \zeta = \frac{\alpha}{\omega_0} = \frac{R}{2L\omega_0} = \frac{1}{2Q_s}$
- **Remember** $\zeta = 1 \Rightarrow Q_s = \frac{1}{2}$
  - **Critical-damped case**

    $$at\ f = f_0 :\ H_c(f) = -jQ_s\ ,\ \left|H_c(f)\right|^2 = Q_s^2 \Rightarrow y = -6dB$$

Notable Points

- Transfer functions for different elements in a circuit
- Phase shift for frequencies in the pass band
- Phase shift for bandpass, low-pass and high-pass
- Meaning of Q

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Lecture 20
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10/17/07

Reading: Chap. 6; Supplementary Notes Ch. 1

Second-Order Band-Pass Filter

$$H_c(f) = \frac{R}{R + \frac{1}{j\omega C} + j\omega L} = \frac{1}{1 + \frac{1}{j\omega RC} + j\omega \frac{L}{R}}$$

$$H_c(f) = \frac{1}{1 + jQ_s\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

$$\left|H_c(f)\right|^2 = \frac{1}{1 + \left[Q_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right]^2}$$

Second-Order Band-Pass Filter

- $Low\ f:\ \frac{f_0}{f}$ dominates $H_c(f) = \frac{1}{-jQ_s\frac{f_0}{f}}; \left|H_c(f)\right|^2 = \frac{1}{\left[Q_s\frac{f_0}{f}\right]^2}$

$$y = 10\log\left|H_c(f)\right|^2 \approx K + 20\log\left|f\right| \Rightarrow Slope: 20dB/dec$$

$$\angle H_c(f) = 90°$$

- $High\ f:\ \frac{f}{f_0}$ dominates $H_c(f) \approx \frac{1}{jQ_s\frac{f}{f_0}},\ \left|H_c(f)\right|^2 \sim \left(\frac{f_0}{Q_s f}\right)^2$

$$y = 10\log\left|H_c(f)\right|^2 \approx K - 20\log f \quad \Rightarrow Slope: -20dB/dec$$

$$\angle H_c(f) = -90°$$

- $at\ f = f_0:\ \ H_c(f) = 1\ ,\ \left|H_c(f)\right|^2 = 1 \Rightarrow y = 0dB$

$$H_c(f) = 0°$$

3dB Point and Bandwidth

Let $f = f_0 + \Delta$, where $\Delta = \delta f_0$, $\frac{f}{f_0} = 1+\delta$, $\frac{f_0}{f} = 1-\delta$

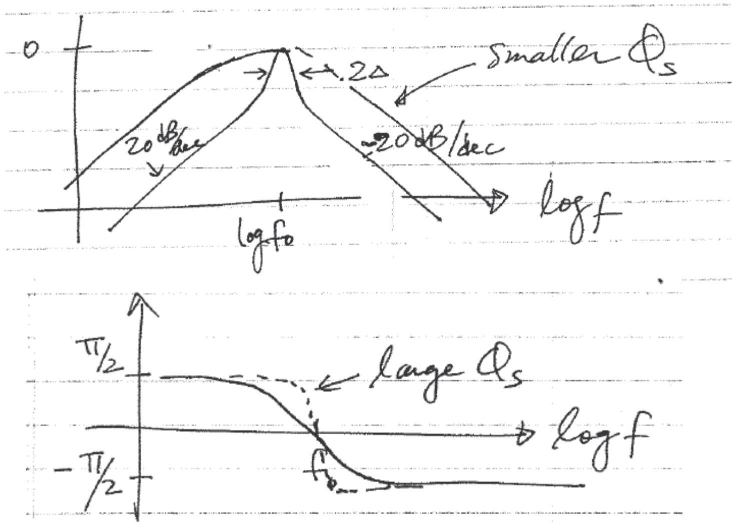
$|H_c(f)|^2$ is 3dB down

$$|H_c(f)|^2 = \frac{1}{2} = \frac{1}{1+\left[Q_s\left(\frac{f}{f_0}-\frac{f_0}{f}\right)\right]^2} = \frac{1}{1+[Q_s(1+\delta-1+\delta)]^2}$$

Hence $Q_s(1+\delta-1+\delta) = 1$

$$2\delta = \frac{1}{Q_s};\ \delta = \frac{1}{2Q_s} = \zeta;\ Bandwidth = 2\Delta = \frac{f_0}{Q_s} = 2\zeta f_0$$

Bode Plots



Parallel LC Circuit

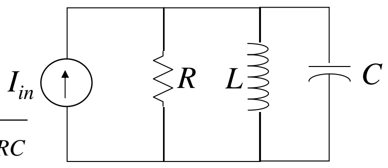
$$Z_p = \frac{1}{\frac{1}{R}+\frac{1}{j\omega L}+j\omega C}$$

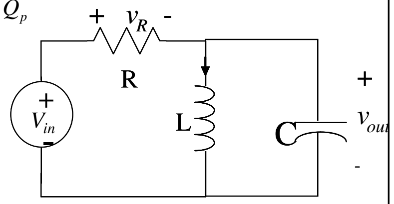
$$H(f) = \frac{I_0 Z_p}{I_0 R} = \frac{1}{\left(\frac{1}{R}+\frac{1}{j\omega L}+j\omega C\right)R} = \frac{1}{1+\frac{R}{j\omega L}+j\omega RC}$$

$$\frac{R}{j\omega L} = \frac{j\omega_0}{\omega}\frac{R}{\omega_0 L} = \frac{j\omega_0}{\omega}Q_p\ ,\quad j\omega RC = j\frac{\omega}{\omega_0}\omega_0 RC = \frac{j\omega}{\omega_0}Q_p$$

$$H_c(f) = \frac{1}{1+jQ_p\left(\frac{f}{f_0}-\frac{f_0}{f}\right)}$$

$$|H_c(f)|^2 = \frac{1}{1+\left[Q_p\left(\frac{f}{f_0}-\frac{f_0}{f}\right)\right]^2}$$





Second-Order Band-Pass Filter

- The derivation is the same as LC in series
- All is the same except Q
- What happens to Bode magnitude and phase plots with increasing Q?