Lecture #9

OUTLINE

– The capacitor
– The inductor

Reading

– Chapter 3

The Capacitor

Two conductors (a,b) separated by an insulator:

difference in potential = V_{ab}

=> equal & opposite charge Q on conductors

$$Q = CV_{ab}$$

(stored charge in terms of voltage)

where C is the **capacitance** of the structure,

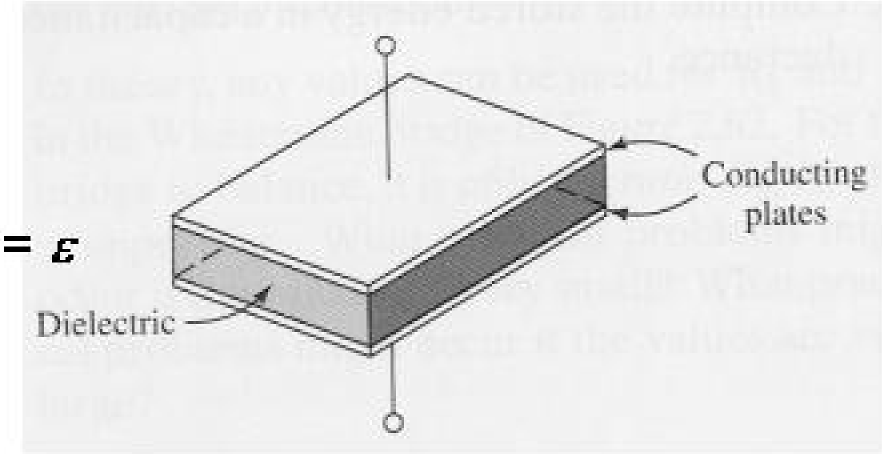
➢ positive (+) charge is on the conductor at higher potential

<u>Parallel-plate capacitor</u>:

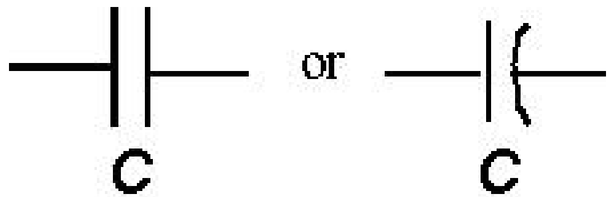
- area of the plates = A
- separation between plates = d
- ***dielectric permittivity*** of insulator = ε

=> capacitance

$$C = \frac{A\varepsilon}{d}$$



Symbol:

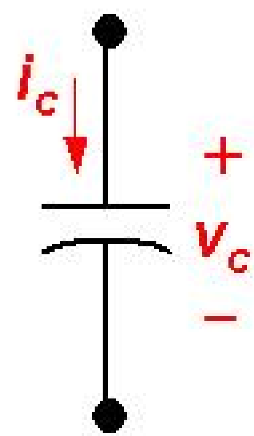


Units: Farads (Coulombs/Volt)

(typical range of values: 1 pF to 1 μF)

Current-Voltage relationship:

$$i_c = \frac{dQ}{dt} = C\frac{dv_c}{dt} + v_c\frac{dC}{dt}$$



Note: Q (v_c) must be a continuous function of time

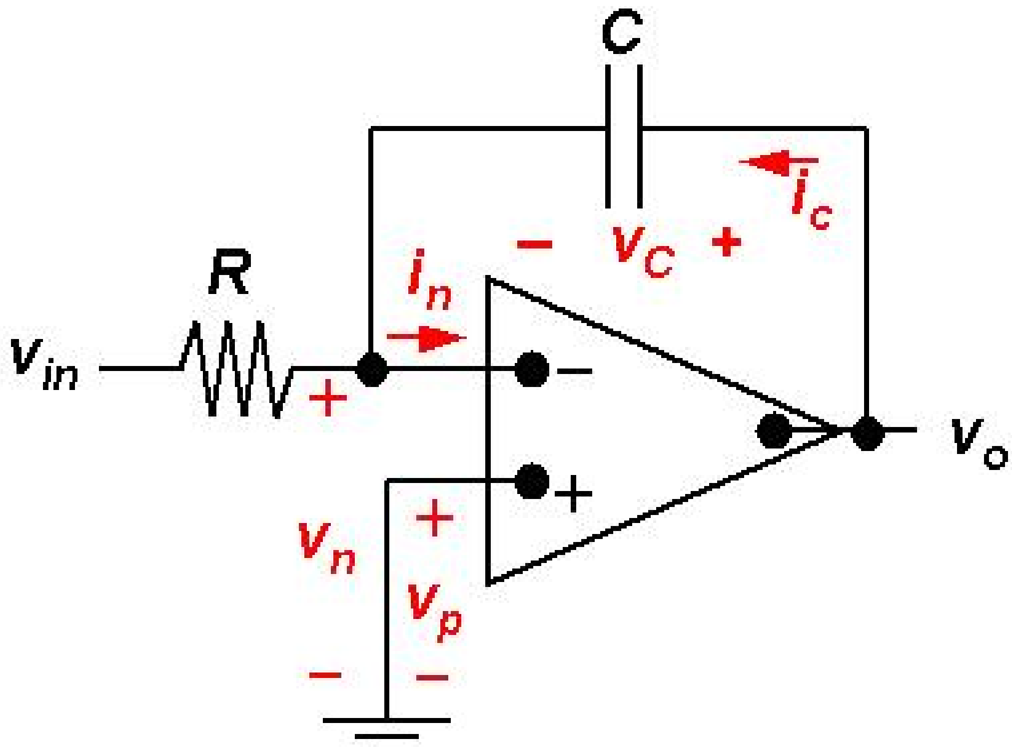
Voltage in Terms of Current

$$Q(t) = \int_0^t i_c(t)dt + Q(0)$$

$$v_c(t) = \frac{1}{C}\int_0^t i_c(t)dt + \frac{Q(0)}{C} = \frac{1}{C}\int_0^t i_c(t)dt + v_c(0)$$

Op-Amp Integrator

$$v_o(t) = -\frac{1}{RC}\int_0^t v_{IN}(t)dt + v_C(0)$$



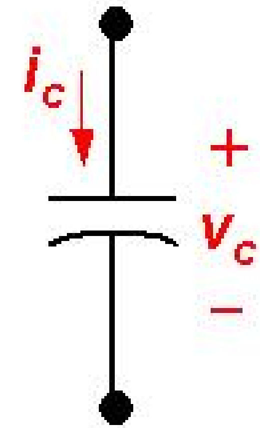
Stored Energy

You might think the energy stored on a capacitor is ***QV***, which has the dimension of Joules. But during charging, the average voltage across the capacitor was only half the final value of ***V*** for a linear capacitor.

Thus, energy is $\frac{1}{2}QV = \boxed{\frac{1}{2}CV^2}$

<u>Example</u>: A 1 pF capacitance charged to 5 Volts has ½$(5V)^2$ (1pF) = 12.5 pJ

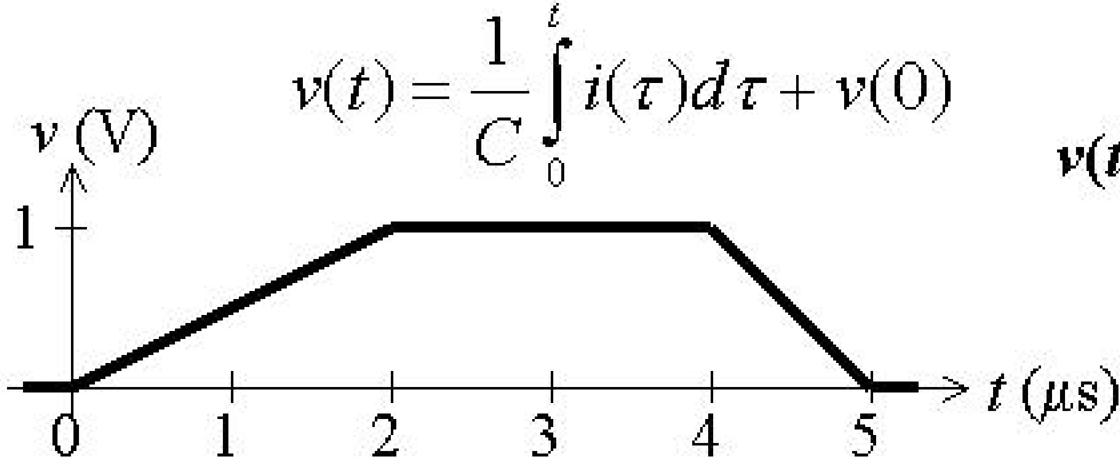
A more rigorous derivation

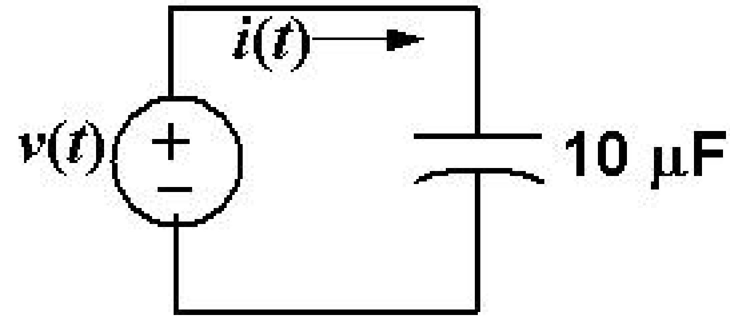


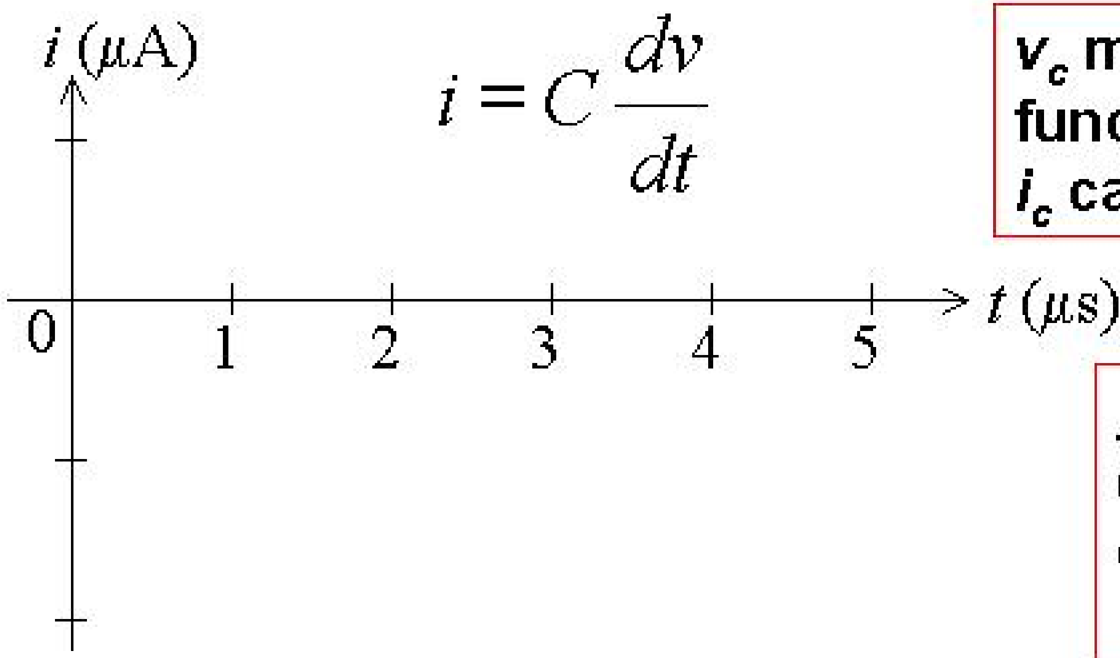
$$w = \int_{t=t_{Initial}}^{t=t_{Final}} v_c \cdot i_c \, dt = \int_{v=V_{Initial}}^{v=V_{Final}} v_c \frac{dQ}{dt} dt = \int_{v=V_{Initial}}^{v=V_{Final}} v_c \, dQ$$

$$w = \int_{v=V_{Initial}}^{v=V_{Final}} C v_c \, dv_c = \frac{1}{2} C V_{Final}{}^{2} - \frac{1}{2} C V_{Initial}{}^{2}$$

Example: Current, Power & Energy for a Capacitor

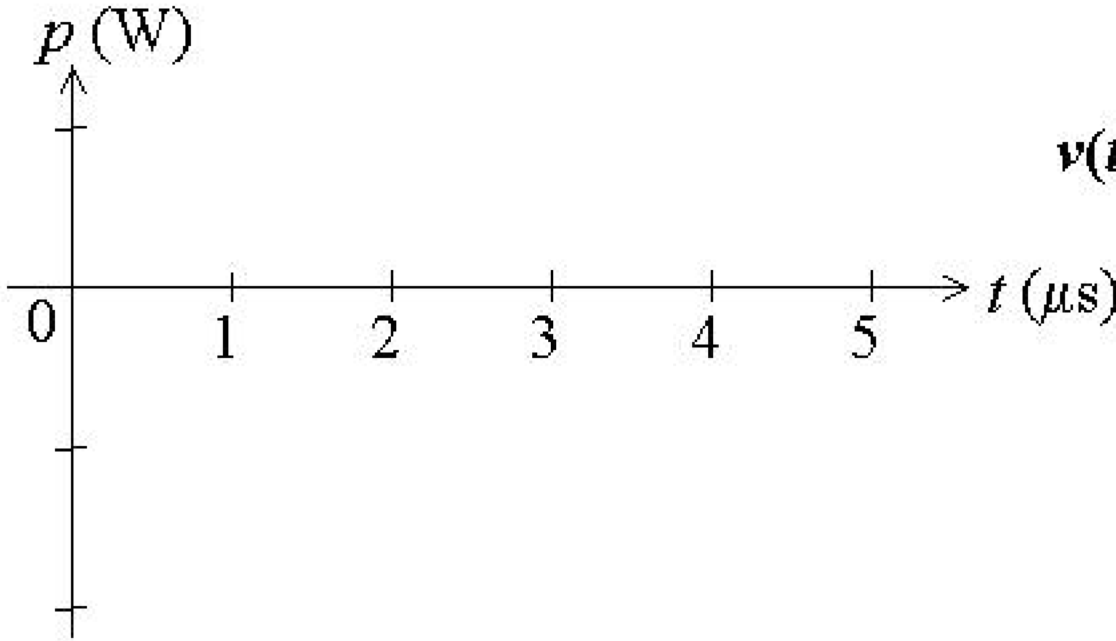


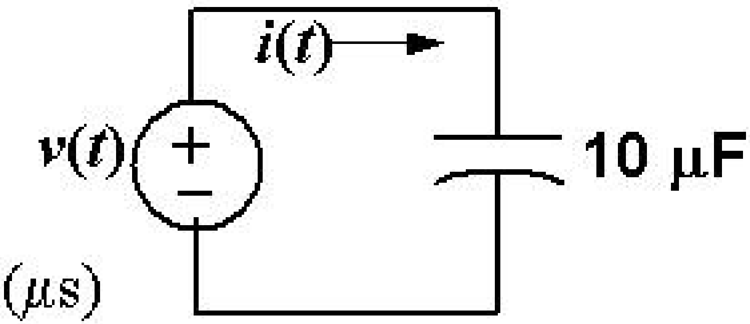




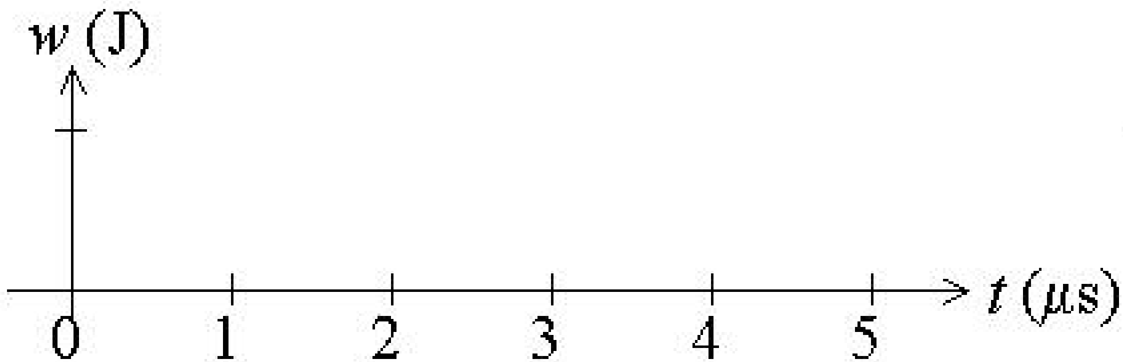
v_c must be a continuous function of time; however, i_c can be discontinuous.

<u>Note</u>: In "steady state" (dc operation), time derivatives are zero → C is an open circuit



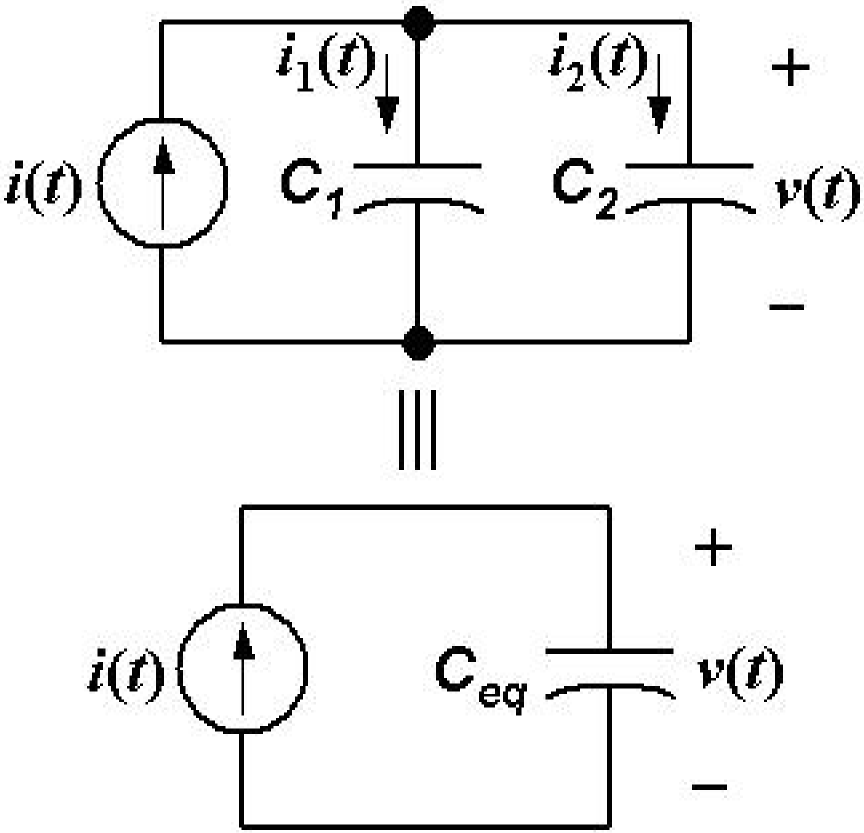


$$p = vi$$



$$w = \int_0^t p\,d\tau = \frac{1}{2} C v^2$$

Capacitors in Parallel

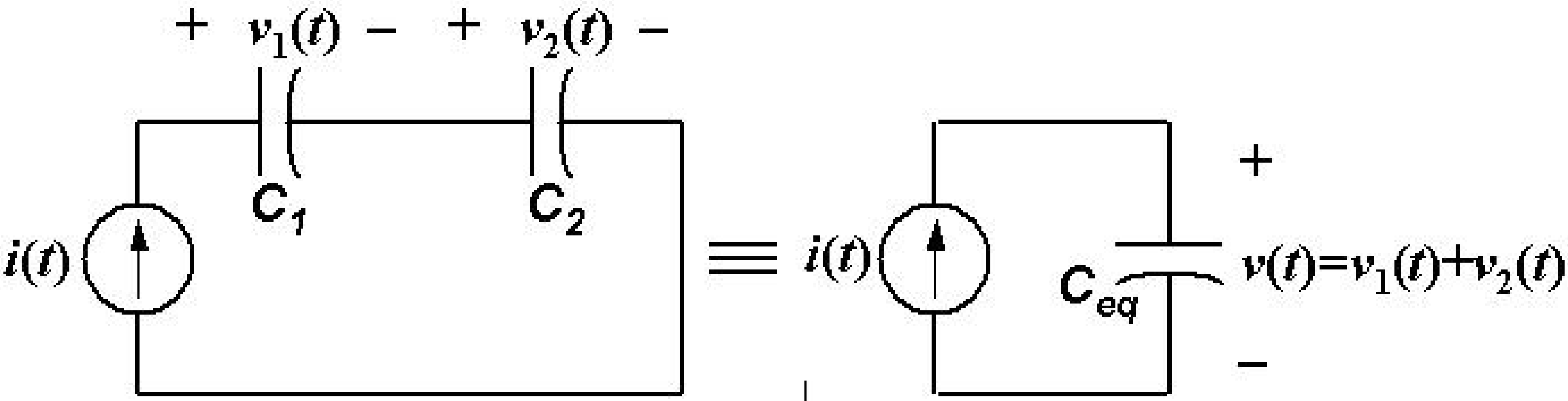


$$i = C_{eq} \frac{dv}{dt}$$

$$C_{eq} = C_1 + C_2$$

Equivalent capacitance of capacitors in parallel is the sum

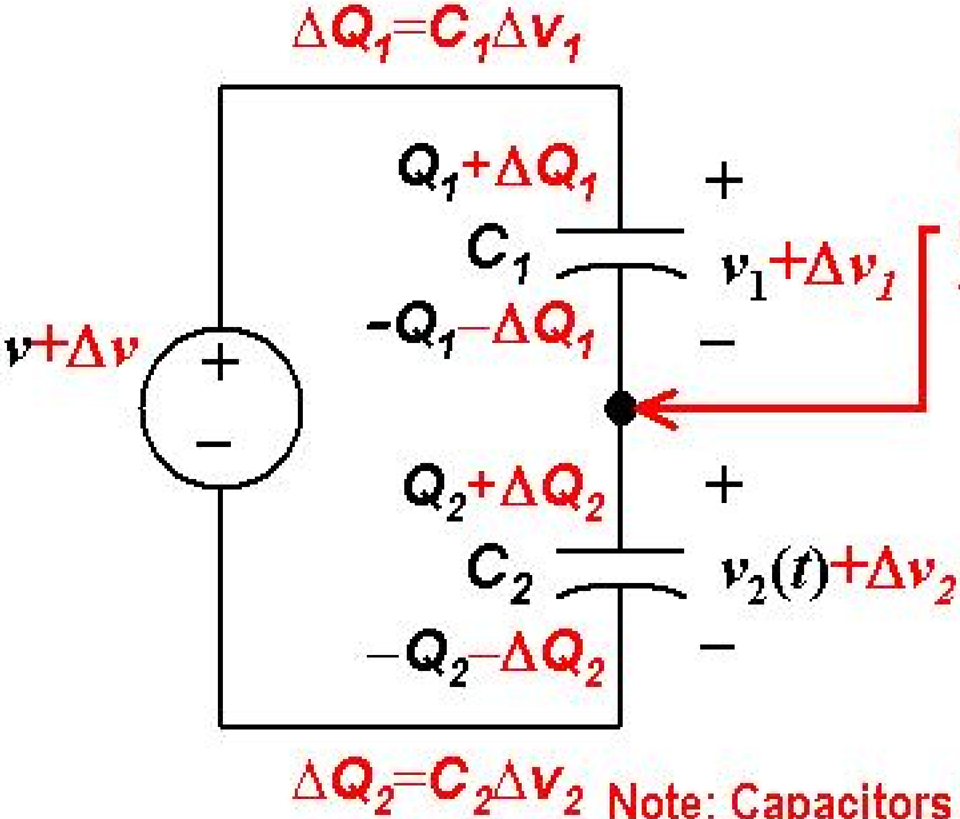
Capacitors in Series



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Capacitive Voltage Divider

Q: Suppose the voltage applied across a series combination of capacitors is changed by Δv. How will this affect the voltage across each individual capacitor?



$$\Delta v = \Delta v_1 + \Delta v_2$$

Note that no net charge can can be introduced to this node. Therefore, $-\Delta Q_1 + \Delta Q_2 = 0$

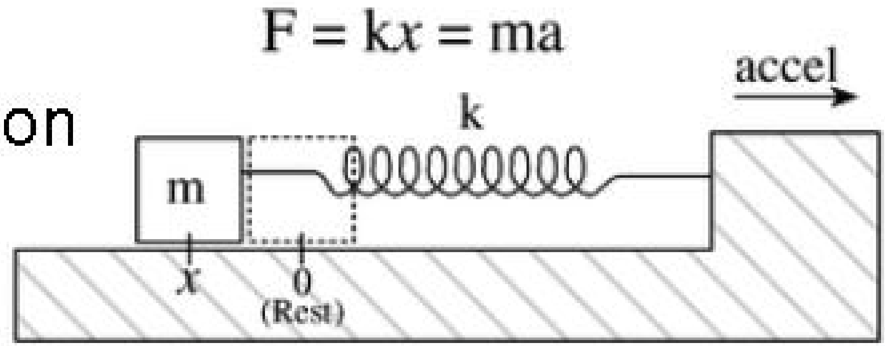
$$\Rightarrow \quad C_1 \Delta v_1 = C_2 \Delta v_2$$

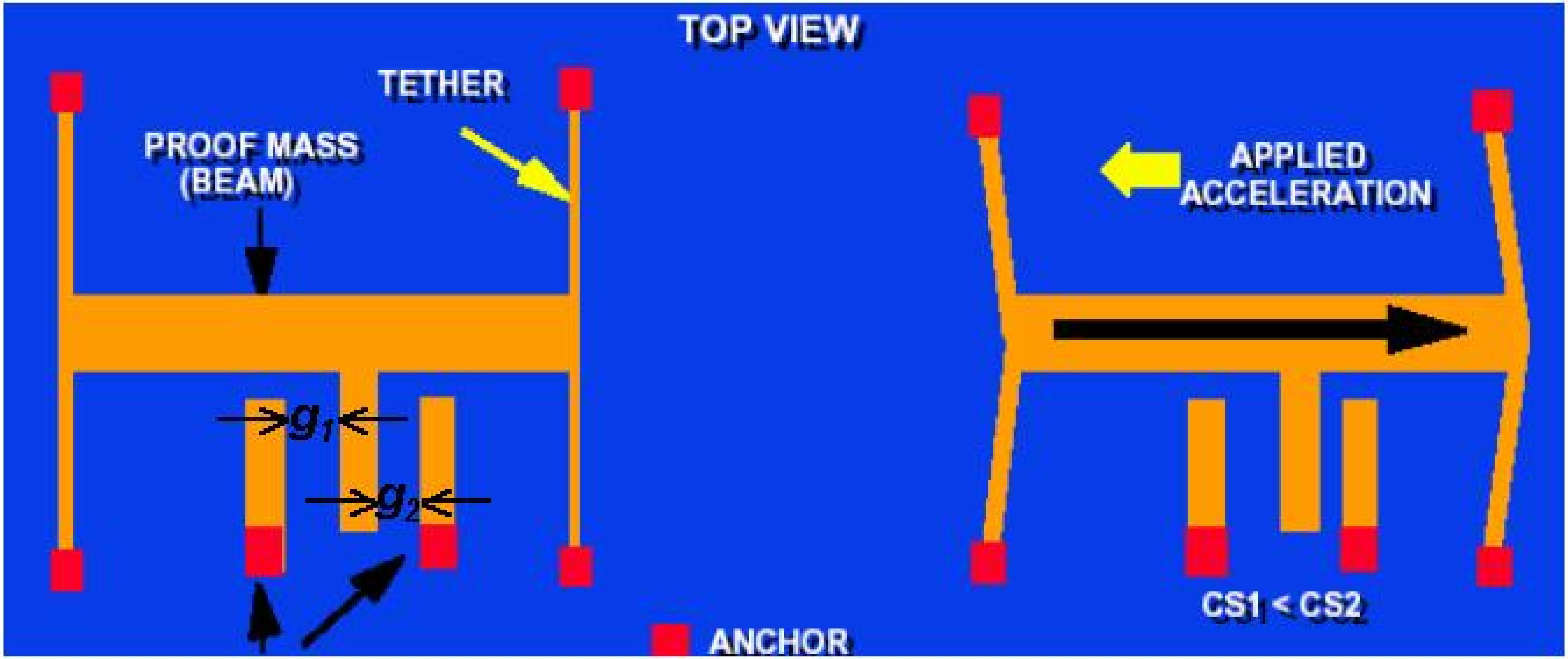
$$\Delta v_2 = \frac{C_1}{C_1 + C_2} \Delta v$$

Note: Capacitors in series have the same incremental charge.

Application Example: MEMS Accelerometer

- Capacitive position sensor used to measure acceleration (by measuring force on a proof mass)



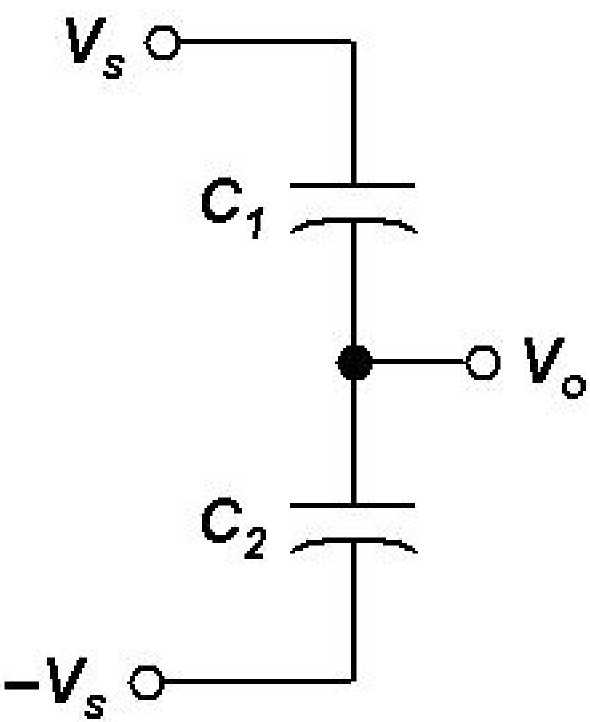


FIXED OUTER PLATES

Sensing the Differential Capacitance

– Begin with capacitances electrically discharged
– Fixed electrodes are then charged to $+V_s$ and $-V_s$
– Movable electrode (proof mass) is then charged to V_o

Circuit model

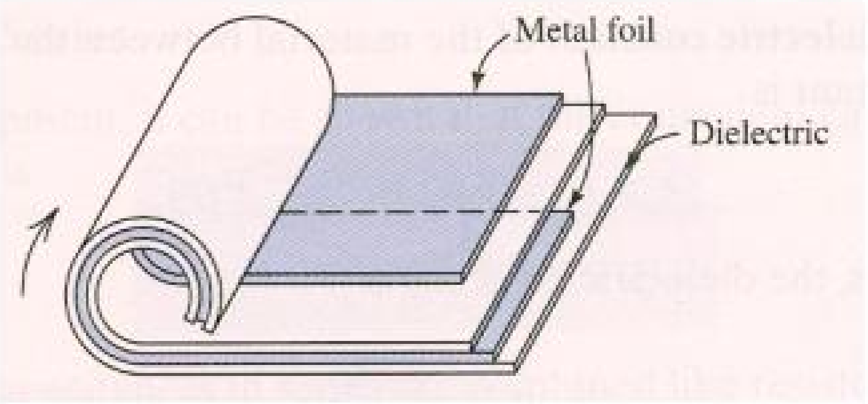


$$V_o = -V_s + \frac{C_1}{C_1 + C_2}(2V_s) = \frac{C_1 - C_2}{C_1 + C_2} V_s$$

$$\frac{V_o}{V_s} = \frac{\frac{\varepsilon A}{g_1} - \frac{\varepsilon A}{g_2}}{\frac{\varepsilon A}{g_1} + \frac{\varepsilon A}{g_2}} = \frac{g_2 - g_1}{g_2 + g_1} = \frac{g_2 - g_1}{const}$$

Practical Capacitors

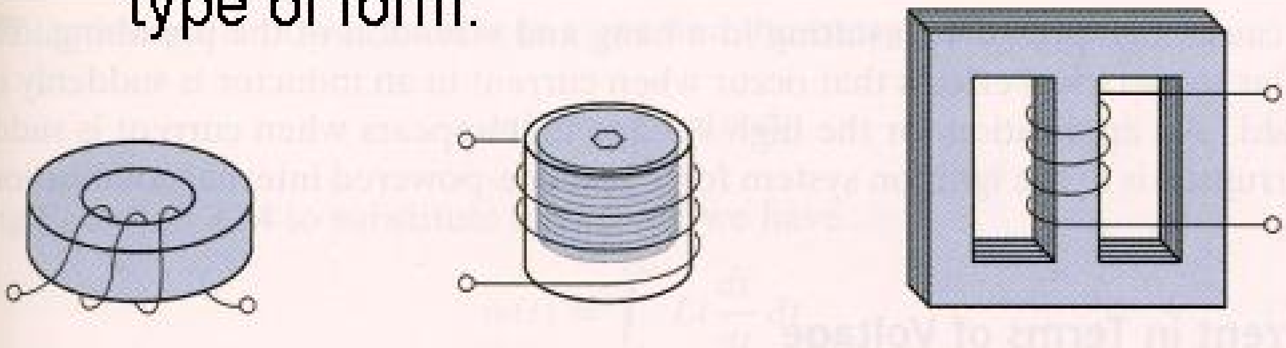
- A capacitor can be constructed by interleaving the plates with two dielectric layers and rolling them up, to achieve a compact size.



- To achieve a small volume, a very thin dielectric with a high dielectric constant is desirable. However, dielectric materials break down and become conductors when the electric field (units: V/cm) is too high.
  - Real capacitors have maximum voltage ratings
  - An engineering trade-off exists between compact size and high voltage rating

The Inductor

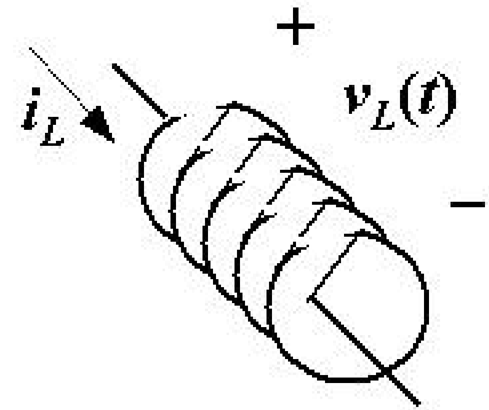
- An inductor is constructed by coiling a wire around some type of form.



(a) Toriodal inductor

(b) Coil with an iron-oxide slug that can be screwed in or out to adjust the inductance

(c) Inductor with a laminated iron core

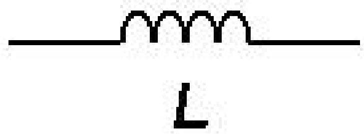


- Current flowing through the coil creates a magnetic field and a magnetic flux that links the coil: Li_L
- When the current changes, the magnetic flux changes → a voltage across the coil is induced:

$$v_L(t) = L\frac{di_L}{dt}$$

<u>Note</u>: In "steady state" (dc operation), time derivatives are zero → *L* is a short circuit

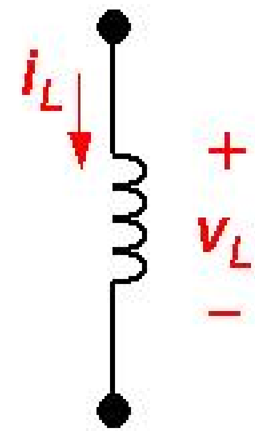
Symbol:



Units: **Henrys (Volts • second / Ampere)**

(typical range of values: μH to 10 H)

Current in terms of voltage:



$$di_L = \frac{1}{L} v_L(t)dt$$

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L(\tau)d\tau + i(t_0)$$

Note: i_L must be a continuous function of time

Stored Energy

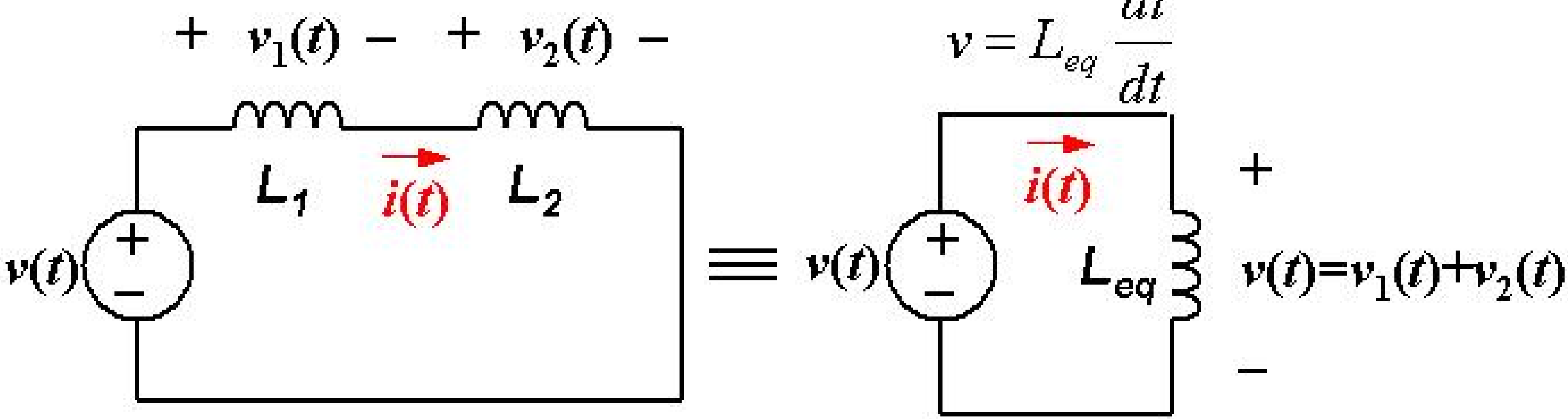
Consider an inductor having an initial current $i(t_0) = i_0$

$$p(t) = v(t)i(t) =$$

$$w(t) = \int_{t_0}^{t} p(\tau)d\tau =$$

$$w(t) = \frac{1}{2}Li^2 - \frac{1}{2}L{i_0}^2$$

Inductors in Series



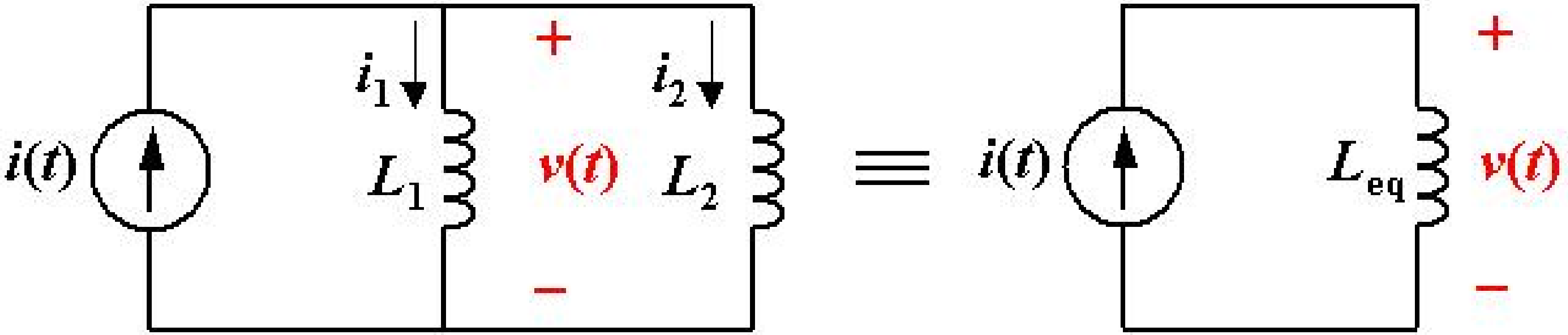
$$v = L_{eq} \frac{di}{dt}$$

$$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} = \left(L_1 + L_2\right)\frac{di}{dt} = L_{eq} \frac{di}{dt}$$

$$\boxed{L_{eq} = L_1 + L_2}$$

Equivalent inductance of inductors in series is the sum

Inductors in Parallel



$$i = \frac{1}{L_{eq}} \int_{t_0}^{t} v\,d\tau + i(t_0)$$

$$i = i_1 + i_2 = \frac{1}{L_1}\int_{t_0}^{t} v\,d\tau + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,d\tau + i_2(t_0)$$

$$i = \left[\frac{1}{L_1} + \frac{1}{L_2}\right]\int_{t_0}^{t} v\,d\tau + \left[i_1(t_0) + i_2(t_0)\right]$$

$$\Rightarrow \boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}} \quad \text{with} \quad i(t_0) = i_1(t_0) + i_2(t_0)$$

Summary

Capacitor	**Inductor**
$i = C\frac{dv}{dt}$	$v = L\frac{di}{dt}$
$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
v cannot change instantaneously	***i*** cannot change instantaneously
i can change instantaneously	***v*** can change instantaneously
Do not short-circuit a charged capacitor (-> infinite current!)	Do not open-circuit an inductor with current (-> infinite voltage!)
n cap.'s in series: $\frac{1}{C_{eq}} = \sum_{i=1}^{n}\frac{1}{C_i}$	*n* ind.'s in series: $L_{eq} = \sum_{i=1}^{n} L_i$
n cap.'s in parallel: $C_{eq} = \sum_{i=1}^{n} C_i$	*n* ind.'s in parallel: $\frac{1}{L_{eq}} = \sum_{i=1}^{n}\frac{1}{L_i}$