EE 40: Introduction to Microelectronic Circuits
Spring 2008: HW 4
Solution

Venkat Anantharam

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Referenced problems from Hambley, 4th edition.

1. P3.39

   The current through the capacitor is

$$i(t) = C\frac{dv_C(t)}{dt} = 10^{-7}(-10^3 \sin(100t)) = -10^{-4}\sin(100t) \text{ A}$$

   (where we assume that $v_C(t)$ is given in volts).

   The voltage across the resistor is therefore

$$v_R(t) = Ri(t) = -10^{-3}\sin(100t)\text{V}.$$

   The concept of 1% accuracy used in the problem statement is not very precise because the voltages are time varying. Interpreting this question as asking whether the peak of the parasitic resistance voltage term is within 1% of the peak voltage across the capacitor, the question may be viewed as asking if

$$10^{-3} < (0.01)10^3.$$

   This is of course true.

   Suppose now that

$$v_C(t) = 0.1\cos(10^7 t)$$

   (again assumed to be given in volts). Then we have

$$i(t) = C\frac{dv_C(t)}{dt} = 10^{-7}(-10^6 \sin(10^7 t)) = -0.1\sin(10^7 t) \text{ A}$$

   and

$$v_R(t) = -\sin(10^7 t) \text{ V}.$$

   In this case the peak of the parasitic resistance voltage is actually 10 times the peak voltage across the capacitor, so it becomes very important to include it in the model.

The message of this problem is that the higher the frequency of the signals the more a capacitor approximates a short circuit (which makes it increasingly inappropriate to neglect parasitic resistances in series).

2. P3.68

The current through the inductor (and resistor) is given as

$$i(t) = 0.1\cos(10^5 t) \text{ A}.$$

Therefore we can find the voltage across the inductor as

$$v_L = L\frac{di(t)}{dt} = 10 * 10^{-3}(-0.110^5 \sin(10^5 t)) = -100\sin(10^5 t)\text{V}.$$

The voltage across the resistor is

$$v_R(t) = Ri(t) = 1 * 0.1\cos(10^5 t) \text{ V},$$

and the total voltage is

$$v(t) = v_L(t) + v_R(t) = -100\sin(10^5 t) + 0.1\cos(10^5 t) \text{ V}$$

The concept of 1% accuracy used in the problem statement is not very precise because the voltages are time varying. Interpreting this question as asking whether the peak of the parasitic resistance voltage term is within 1% of the peak voltage across the capacitor, the question may be viewed as asking if

$$0.1 < (0.01)100.$$

This is of course true.

Suppose now that

$$i(t) = 0.1\cos(10t)$$

(again assumed to be given in Amps). Then we have

$$v_L = L\frac{di(t)}{dt} = 10 * 10^{-3}(-0.1 * 10\sin(10t)) = -10^{-2}\sin(10^5 t) \text{ V},$$

while $v_R(t) = Ri(t) = 0.1\cos(10t)$.

In this case the peak of the parasitic resistance voltage is actually 10 times the peak voltage across the inductor, so it becomes very important to include it in the model.

The message of this problem is that the **lower** the frequency of the signals the more an inductor approximates a short circuit (which makes it increasingly inappropriate to neglect parasitic resistances in series).

3. P3.71

$$\begin{aligned} i(t) &= C\frac{dv_C(t)}{dt} = 5*10^{-4}*1000*10\cos(1000t) = 5\cos(1000t) \\ v_L(t) &= L\frac{di_L(t)}{dt} = -2*10^{-3}*1000*5\sin(1000t) = -10\sin(1000t) \\ v(t) &= v_C(t)+v_L(t) = 10\sin(1000t) - 10\sin(1000t) = 0 \end{aligned}$$

This is an idealized LC oscillator with no active source, where energy is sloshing back and forth between the capacitor and the inductor.

4. P4.4
Writing KVL after line 0 we have

$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_s \text{ for } t \geq 0.$$

The solution comes as a sum of two parts. One is the **particular solution**. This is **any solution** of the differential equation for $t \geq 0$ where we do not worry about matching the initial conditions. Thus the notion of "particular solution" is not unique.

The second part is the **complementary solution**. We get this by finding a solution of the **homogenous equation**

$$RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$

corresponding to an initial condition which is the sum of the true initial condition and a part which compensates for the initial condition in the particular solution.

For instance, in this case we could observe that

$$v_C^{part}(t) = V_s \text{for } t \geq 0$$

is a valid choice of particular solution and it satisfies the differential equation. However, its initial value is V_s so to find the corresponding complementary solution we would need to find the solution to the homogeneous equation from the initial condition $(V_C(0_-) - V_s)$ which is

$$V_C^{comp}(t) = (v_C(0_-) - V_s)e^{-t/RC}, t \geq 0$$

Alternately, we might have observed that

$$v_C^{part}(t) = V_s - V_s e^{-t/RC}, t \geq 0$$

is also a valid particular solution. In this case there is no need to compensate for the initial conditions of this solution in the complementary solution. The corresponding complementary solution would here be

$$V_C^{comp}(t) = v_C(0_-)e^{-t/RC}, t \geq 0$$

In either case, the overall solution is

$$v_C(t) = v_C^{part}(t) + v_C^{comp}(t) = v_s + (v_C(0_-) - V_s)e^{-t/RC}\text{for } t \geq 0$$

Substituting the given values, we have

$$RC = 10^5 * 10^{-8} = 10^{-3}\text{seconds}$$

and

$$v_C(t) = 100 - 150e^{-1000t}\text{volts, } t \geq 0$$

where t is measured in seconds.

5. 4.9

From KVL just before the switch opens we know that

$$v_c(0_-) = v(0_-) = 0$$

This is because the voltage across a short circuit must be zero.

We are interested in $v(t)$ for $t \geq 0$, ie after the switch opens. Writing I_s for the current source (for clarity) this is governed by the differential equation

$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = I_s,\ t \geq 0$$

which we get by writing KCL after the switch opens.

The solution in general is the sum of two parts. The first part is the particular solution, which in this case we can take as

$$v^{part}(t) = RI_s - RI_s e^{-t/RC}\ ,\ t \geq 0$$

The second part is the complementary solutions which for this differential equation would in general be, for the given particular solution,

$$v^{comp}(t) = v(0_-)e^{-t/RC}\ ,\ t \geq 0$$

but in this example we know from the conditions before the switch opened that $v(0_-) = 0$, as we have

$$v^{comp}(t) = 0,\ t \geq 0$$

The solution is therefore

$$v(t) = v^{part}(t) + v^{comp}(t) = RI_s - RI_s e^{-t/RC}\ ,\ t \geq 0$$

Substituting the given values, we have

$$RC = 10^4 * 10^{-6} = 10^{-2}\text{secs}$$

and

$$v(t) = 10 - 10e^{-100t}\ ,\ t \geq 0$$

where t is measured in seconds.

6. P4.17

(a) Writing KVL immediately after the switch closes gives

$$v_1(0_+) = 10^5 i(0_+) + v_2(0_+).$$

Since the voltage across a capacitor cannot jump we have

$$v_1(0_+) = v_1(0_-) = 100\text{V}$$

and

$$v_2(0_+) = 1v_2(0_-) = 0\text{V}.$$

Substituting, we get

$$10^5 i(0_+) = 100$$
$$i(0_+) = 10^{-3}\text{amps}.$$

(b) For times $t \geq 0$ we may write KVL to get

$$v_1(t) = Ri(t) + v_2(t).$$

Taking the derivative, we have

$$\frac{dv_1(t)}{dt} - \frac{dv_2(t)}{dt} - R\frac{di(t)}{dt} = 0.$$

But we also have

$$-i(t) = C_1\frac{dv_1(t)}{dt}$$

and

$$i(t) = C_2\frac{dv_2(t)}{dt}.$$

Hence, the differential equation can be written as

$$R\frac{di(t)}{dt} + (\frac{1}{C_1} + \frac{1}{C_2})i(t) = 0.$$

This is a homogeneous differential equation corresponding to the absence of sources. Note that the capacitors in series appear to have been replaced, in effect, by a single capacitor of value $1/(\frac{1}{C_1} + \frac{1}{C_2})$.

(c) The time constant of the circuit would be RC where

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}.$$

Substituting the given values, we have

$$C = 0.5\mu F$$
$$RC = 10^5 * 0.5 * 10^{-6} = 0.05\text{secs}.$$

(d) Since the governing differential equation is homogeneous the particular solution is zero for all $t \geq 0$. We need only consider the complementary solution, which is

$$i(t) = i(0_+)e^{-t/RC} \text{for } t \geq 0.$$

Substituting the values calculated earlier, we get

$$i(t) = 10^{-3}e^{-20t} \text{for } t \geq 0$$

where $i(t)$ is in amps and t is in seconds.

(e) From the defining equation for the capacitors C_2 we may write, for $t \geq 0$,

$$v_2(t) = v_2(0_-) + \frac{1}{C_2}\int_0^t i(s)ds$$

Substituting, this gives

$$v_2(t) = 10^6 \int_0^t 10^{-3}e^{-20s}ds = \frac{10^3}{20}(1 - e^{-20t}) = 50(1 - e^{-20t})\text{volts}$$

This gives

$$\lim_{t\to\infty} v_2(t) = 50\text{volts}$$

as the limiting value of the voltage across the capacitor C_2.
Note that

$$\lim_{t\to\infty} i(t) = 0$$

Thus in the limit as t becomes very large the voltages across the capacitor C_1 and the capacitor C_2 should be the same. This can be verified by writing

$$v_1(t) = v_1(0_-) - \frac{1}{C_1}\int_0^t i(s)ds$$

and substituting to get

$$v_1(t) = 100 - 10^6 \int_0^t 10^{-3}e^{-20s}ds = 100 - \frac{10^3}{20}(1 - e^{-20t}) = 50 + 50e^{-20t}\text{volts.}$$

So

$$\lim_{t\to\infty} v_1(t) = 50\text{volts}$$

Note that you could also have solved this part of the problem by using the principle of conservation energy. The initial energy stored in the capacitor C_2 is zero. The total energy dissipated in the resistor over all $t \geq 0$ is

$$\int_0^\infty Ri^2(t)dt = \int_0^\infty 10^5 * 10^{-6}e^{-40t}dt = \frac{10^-1}{40} = 1/400\text{Joules.}$$

Thus in the limit as $t \to \infty$,

$$\frac{1}{200} - \frac{1}{400} = \frac{1}{400} J$$

must remain stored in the capacitors. Since each of them must have the same voltage across them and they are both of the same capacitance, each must be storing $1/800J$ in the limit as $t \to \infty$. Solving for v in the equation

$$Cv^2/2 = 1/800$$

with $C = 10^{-6}$Farads gives

$$v = 50\text{Volts}$$

as expected.

Yet another approach would be to say that the positive charge stored on the upper plates of the capacitors can't cross to the bottom plates. Therefore $Q_1(t) + Q_2(t) = Q_tot(0_-)$. Further, we have a resistor, and so energy is dissipated until the system settles in the minimum energy configuration. So, minimizing the energy, which is $\frac{1}{2}C_1Q_1^2 + \frac{1}{2}C_2Q_2^2$, subject to the charge conservation constraint, we get $Q_1 = Q_2 = Q_tot(0_-)/2$. Therefore $Q_1 = 50$V which is what we found before.

7. P4.37

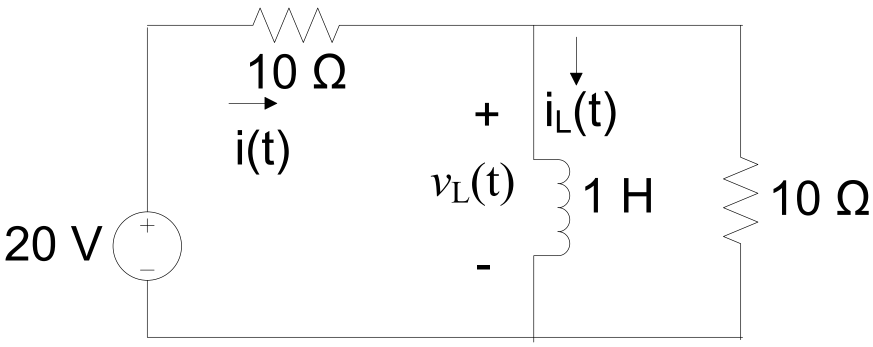

Figure 1: Circuit for P4.37

For times $t \leq 0$ applying KVL gives

$$i(t) = 20/20 = 1A.$$

After the switch closes (i.e. for $t \geq 0$) we have the circuit where we have also indicated our choice of reference for the voltage $v_L(t)$ across the inductor and for the current $i_L(t)$ through the inductor.

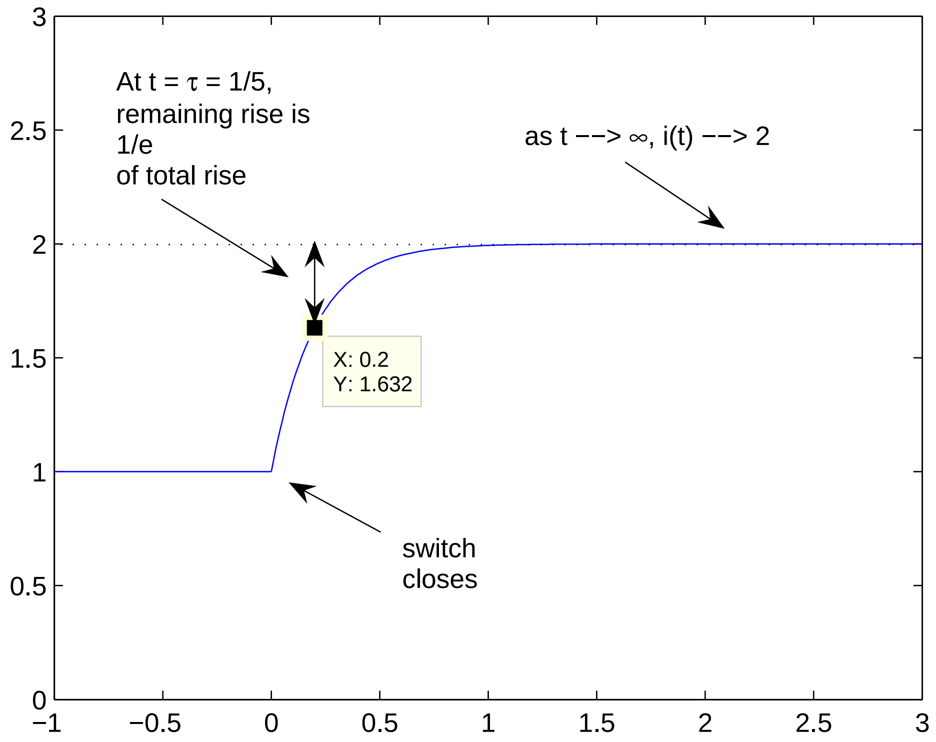

Figure 2: Plot of i(t) for P4.37

Note that

$$\lim_{t\to\infty} i(t) = 2\text{A}.$$

The current through an inductor cannot change suddenly. Thus we have

$$i_L(0_+) = i_L(0_-) = 0.$$

Writing KCL, we have

$$i(t) = i_L(t) + v_L(t)/10 \text{ for } t \geq 0$$

(Here we also implicitly used one KVL equation). Writing the other KVL equation, we have

$$20 = 10i(t) + v_L(t) \text{ for } t \geq 0$$

Substituting for i(t) from the preceding equation, we get

$$20 = 10i_L(t) + 2v_L(t) \text{ for } t \geq 0$$

Since we also have

$$v_L(t) = L\frac{di_L(t)}{dt} = \frac{di_L(t)}{dt}$$

we finally get

$$\frac{di_L(t)}{dt} + 5i_L(t) = 10 \text{ for } t \geq 0$$

as the differential equation characterizing the circuit.

This equation has the particular solution

$$i_L^{part}(t) = 2 - 2e^{-5t} \text{ , } t \geq 0$$

and the complementary solution in general looks like

$$i_L^{comp}(t) = i_L(0_-)e^{-5t} \text{ for } t \geq 0.$$

Here we have$i_L(0_-) = 0$. Hence the solution to the differential equation is

$$i_L(t) = i_L^{part}(t) + i_L^{comp}(t) = 2 - 2e^{-5t} \text{ for } t \geq 0.$$

From this we get

$$v_L(t) = \frac{di_L(t)}{dt} = 10e^{-5t} \text{ for } t \geq 0$$

which gives

$$i(t) = i_L(t) + v_L(t)/10 = 2 - 2e^{-5t} + e^{-5t} = 2 - e^{-5t} \text{ for } t \geq 0.$$

We plot $i(t)$ in Fig. 2:

7. P4.45 The system is described by the differential equation

$$RC\frac{dv_C(t)}{dt} = v_C(t) = v(t) \text{ for } t \geq 0.$$

As suggested, since $v(t) = t$ for $t \geq 0$, we look for a particular solution of the form

$$v_{cp}(t) = A + Bt \text{ for } t \geq 0.$$

Substituting this into the differential equation gives

$$RCB + A + Bt = t, \; t \geq 0.$$

This can be satisfied by choosing

$$B = 1 \text{ and } A = -RC.$$

We therefore work with the particular solution

$$v_C^{part}(t) = -RC + t, \; t \geq 0.$$

Note that this equals $-RC$ at time 0.

Thus the corresponding complementary solution has to compensate for this initial condition where the true initial condition is given as $v_c(0) = 0$, the complementary solution would be the solution of the homogeneous equation

$$RC\frac{dv_C(t)}{dt} + v_C(t) = 0 \ , \ t \geq 0,$$

from the initial condition RC, i.e.

$$v_C^{comp}(t) = RCe^{-t/RC} \ , \ t \geq 0.$$

The overall solution is then

$$v_C(t) = v_C^{part}(t) + v_C^{comp}(t) = t - RC(1 - e^{-t/RC}) \ , \ t \geq 0.$$

A sketch of this solution $v_c(t)$ is given in Fig. 3.

8. P4.50

By writing KVL after the switch closes we get the differential equation describing the current as

$$2\frac{di_L(t)}{dt} + i_L(t) = 5e^{-t}\sin(t) \ , \ t \geq 0$$

where we used the given values $L = 2H$ and $R = 1\Omega$.

We seek a particular solution of the form

$$Ae^{-t}\sin(t) + Be^{-t}\cos(t) \text{ for } t \geq 0.$$

Substituting this into the equation gives

$$2(-Ae^{-t}\sin(t) + Ae^{-t}\cos(t) - Be^{-t}\cos(t) - Be^{-t}\sin(t)) + (Ae^{-t}\sin(t) + Be^{-t}\cos(t).$$

This is satisfied by the choices $A = -1$ and $B = -2$. We therefore take as particular solution

$$i_L^{part}(t) = -e^{-t}\sin(t) - 2e^{-t}\cos(t), \ t \geq 0.$$

Note that this choice has $i_L^{part}(0) = -2$. The corresponding complementary solution is the solution of the homogenous equation

$$2\frac{di_L(t)}{dt} + i_L(t) = 0$$

from the initial condition 2, because the true initial condition must be $i_L(0_-) = 0$ (prior to the switch closing there cannot be any current flowing through the inductor).

Thus we have

$$i_L^{comp}(t) = 2e^{-t/2}, \ t \geq 0.$$

The overall solution is therefore

$$i_L(t) = i_L^{part}(t) + i_L^{comp}(t) = 2e^{-t/2} - e^{-t}\sin(t) - 2e^{-t}\cos(t), \ t \geq 0.$$

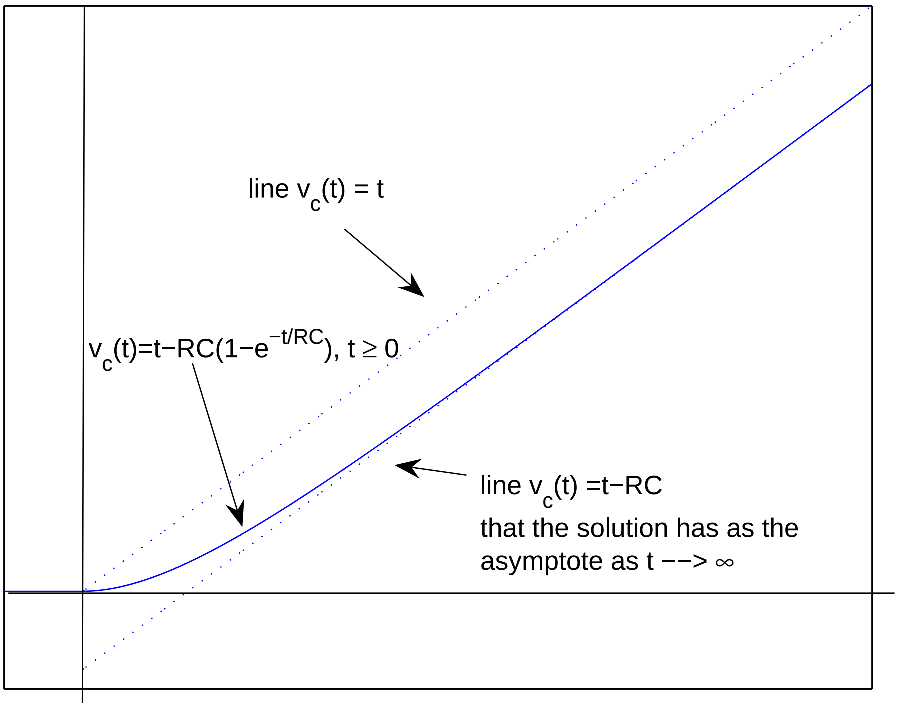

Figure 3: Plot of $v_C(t)$ for P4.45

9. P4.22

   We are interested in a DC steady-state analysis. In DC steady-state the inductor acts as a short circuit and the capacitor acts as an open circuit. Thus

$$i_2 = 100/10^3 = 0.1A \quad (1)$$
$$i_3 = 0A \quad (2)$$
$$i_4 = 100/10^3 = 0.1A \quad (3)$$

   and $i_1 = i_2 + i_3 + i_4 = 0.2A$.