EE 40: Introduction to Microelectronic Circuits
Spring 2008: HW 5
(due 3/7, 5 pm)

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Referenced problems from Hambley, 4th edition.

1. P4.27
   The driving sources are DC, so the circuit will be in DC steady state. Hece, we can replace the capacitor by an open circuit and the inductor by a short circuit, getting the circuit depicted in Figure 5.
   Applying KCL, we have

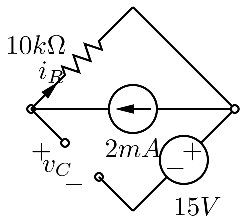

Figure 1: Circuit 2

$$i_R = 2mA$$

Applying KVL, we have

$$v_c = 2mA * 10k\Omega + 15V = 20V + 15V = 35V$$

2. P4.62 *(NOTE: The is a typo in part (b) of the problem P4.61, which should say $v'(0+) = 10^9 \frac{V}{s}$.)*
   (a) The differential equation describing the current after the switch opens is (see equation (4.102)):

$$\frac{d^2v(t)}{dt} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0 \tag{1}$$

From equation (4.104), the undamped resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^7 \frac{1}{s}$$

From equation (4.103), the damping coefficient is

$$\alpha = \frac{1}{2RC} = 10^7 \frac{1}{s}$$

From equation (4.71), the damping ratio is

$$\zeta = \frac{\alpha}{\omega_0} = 1$$

Note that ζ is dimensionless.

(b) To solve the differential equation (1), we need the initial conditions $v(0+)$ and $\frac{dv(0+)}{dt}$. We are given that $v(0+) = 0$ and $i_L(0+) = 0$. (These come from $v(0-) = 0$ and $i_L(0-) = 0$ and physics contraints that $v(0-) = v(0+)$ and $i_L(0-) = i_L(0+)$.)
we can find $\frac{dv(0+)}{dt}$ by determining $i_c(0+)$ (where we use a positive reference). Since $v(0+) = 0$, no current flows through the resistor at 0+, and likewise since $i_L(0+) = 0$, no current flows through the inductor at 0+. Hence,

$$i_c(0+) = 1A$$

This gives

$$C\frac{dv(0+)}{dt} = 1A$$

Therefore,

$$\frac{dv(0+)}{dt} = 10^9\frac{V}{s}$$

(c) Since the equation has zero forcing function, we can simply take

$$v^p(t) = 0$$

as a particular solution.

(d) Since $\zeta = 0$, we are in the critically damped case. From equation (4.75), we see that the complementary solution has the form

$$v^c(t) = K_1e^{-\alpha t} + K_2te^{-\alpha t}$$

because $\omega_n = \sqrt{\omega_o^2 - \alpha^2} = 0$ and so we have $s_1 = s_2 = -\alpha$ (see equations (4.72), (4.73), and (4.76)). Here, K_1 and K_2 are constants that should be chosen so that the total solution

$$v(t) = v^p(t) + v^c(t)$$

satisfies the initial conditions.
This means

$$0 = v(0+) = K_1$$

and

$$10^9\frac{V}{s} = \frac{dv(0+)}{dt} = -\alpha K_1 + K_2$$

This results in the final result

$$v(t) = 10^9te^{\frac{10^7}{s}t}\frac{V}{s}$$

3. P4.66
Treating $i(t)$ as the basic variable, the differential equation describing the circuit after the switch is closed can be written as (see equation (4.59))

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{d}{dt}10\cos{(100\frac{t}{s})}$$

Substituting for the given values, this becomes

$$\frac{d^2i(t)}{dt^2} + 400\frac{di(t)}{dt} + 10^4i(t) = -10^3\sin{(100\frac{t}{s})}$$

The undamped resonant frequency (see equation (4.61)) is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^2\frac{1}{s}$$

The damping coefficient (see eqn (4.60)) is:

$$\alpha = \frac{R}{2L} = 200\frac{1}{s}$$

The daming ratio (see equation (4.71)) is

$$\zeta = \frac{\alpha}{\omega_0} = \frac{200}{100} = 2$$

Since $\zeta > 1$, the circuit is overdamped.
We first find a particular solution of the form

$$i^p(t) = A\cos(100\frac{t}{s}) + B\sin(100\frac{t}{s})$$

By substituting into the differential equation, we get

$$\begin{aligned} -10^4 A\cos(100\frac{t}{s}) - 10^4 B\sin(100\frac{t}{s}) & \\ -4 * 10^4 A\sin(100\frac{t}{s}) + 4 * 10^4 B\cos(100\frac{t}{s}) & \\ +10^4 A\cos(100\frac{t}{s}) + 10^4 B\sin(100\frac{t}{s}) & = -10^3\sin(100\frac{t}{s}) \end{aligned}$$

which requires that $B = 0$ and $A = \frac{1}{40}$.
The general solution to the homogeneous equation

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0$$

is of the form

$$K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

where

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega^2}$$

(see equations (4.74), (4.72), and (4.73)). Both s_1 and s_2 are real because the circuit is overdamped.
Substituting the calculated values gives

$$s_1 = 10^2(-2 + \sqrt{3})$$

$$s_2 = 10^2(-2 - \sqrt{3})$$

The constants K_1 and K_2 in the complementary solution

$$i^c(t) = K_1 e^{s_1 \frac{t}{s}} + K_2 e^{s_2 \frac{t}{s}}$$

should be chosen so that

$$i(t) = i^p(t) + i^c(t) = \frac{1}{40}\cos(100\frac{t}{s} + K_1 e^{-100(2-\sqrt{3})\frac{t}{s}} + K_2 e^{-100(2+\sqrt{3})\frac{t}{s}} \tag{2}$$

matches the initial conditions

$$i(0+) = 0$$

(which comes because the current through the inductor cannot jump) and

$$\frac{di(0+)}{dt} = 10$$

(which comes because at time 0, the entire initial voltage of 10V has to drop accross the inductor).
This gives

$$\frac{1}{K_1} + K_1 + K_2 = 0$$

$$-100(2 - \sqrt{3})K_1 - 100(2 + \sqrt{3})K_2 = 10$$

Solving this system of linear equations yields

$$K_1 = 0.00193$$

$$K_1 = -0.02693$$

Plugging this into (2) yields the final result.

$$i(t) = i^p(t) + i^c(t) = \frac{1}{40}\cos(100\frac{t}{s} + 0.00193e^{-100(2-\sqrt{3})\frac{t}{s}} + -0.02693e^{-100(2+\sqrt{3})\frac{t}{s}}$$

4. P5.16 Assuming that the time axis is in seconds, the period of the waveform is $T = 1s$. Hence,

$$\begin{aligned} V^{\text{RMS}} &= \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} \\ &= \sqrt{\frac{1}{T}\int_0^T (3e^{-1}V)^2 dt} \\ &= \sqrt{\int_0^1 (3e^{-1}V)^2 dt} \\ &= \sqrt{9\int_0^1 e^{-2}dtV} \\ &= \sqrt{\frac{9}{2}[1-e^{-2}]V} \end{aligned}$$

Note that the scaling of the time axis could have been changed without affecting this calculation.

5. Suppose that $v_1(t) = 80\cos(\omega t)$ and $v_2(t) = 60\sin(\omega t)$. Use phasors to reduce the sum $v_s(t) = v_1(t) + v_2(t)$ to a single term of the form $V_m\cos(\omega t + \theta)$. Draw a phasor diagram, showing $\mathbf{V_1}$, $\mathbf{V_2}$, and $\mathbf{V_s}$. State the phase relationships between each pair of these phasors.
The phasor corresponding to

$$v_1(t) = 80\cos(\omega t)$$

is

$$\mathbf{V_1} = 80\angle 0 = 80$$

Equivalently, we get for

$$v_2(t) = 60\sin(\omega t) + 60\cos(\omega t - \frac{\pi}{2})$$

the phasor

$$\mathbf{V_2} = 60\angle -\frac{\pi}{2} = -j60$$

To find the sum using phasors we add the complex numbers which yields

$$\mathbf{V} = \mathbf{V_1} + \mathbf{V_2} = 80 - j60 = 100e^{-j\arctan\frac{3}{4}}$$

This corresponds to the time signal

$$v_s(t) = 100\cos(\omega t - j\arctan\frac{3}{4})$$

The phasor diagram is depicted in Figure 2.
Note that $\mathbf{V_1}$ leads $\mathbf{V_2}$ by $\frac{\pi}{2}$ and leads $\mathbf{V_s}$ by $\arctan(\frac{4}{3})$.

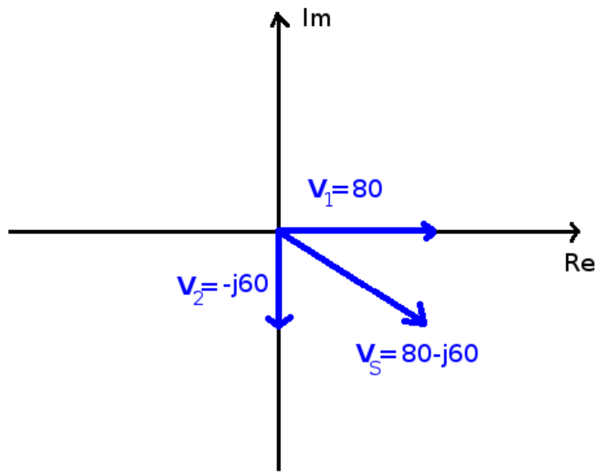

Figure 2: Phasor Diagram

$v_1(t)$	$20\sin(\omega t)$	$20\angle -\frac{\pi}{2}$	$0 - j20$
$v_2(t)$	$20\cos(\omega t + \frac{\pi}{6})$	$20\angle \frac{\pi}{6}$	$10\sqrt{3} + j10$
$v_3(t)$	$20\sin(\omega t + \frac{\pi}{3})$	$20\angle -\frac{\pi}{6}$	$10\sqrt{3} - j10$
$v_4(t)$	$-10\cos(\omega t)$	$20\angle -\pi$	$-10 + j0$

Table 1: Time Signals and their corresponding Phasors

6. Find an expression for $v(t)$ of the form $V_m \cos(\omega t + \theta)$ when $v(t) = v_1(t) + v_2(t) + v_3(t) + v_4(t)$ with

$$\begin{aligned} v_1(t) &= 20\sin(\omega t) \\ v_2(t) &= 20\cos(\omega t + \frac{\pi}{6}) \\ v_3(t) &= 20\sin(\omega t + \frac{\pi}{3}) \\ v_4(t) &= -10\cos(\omega t) \end{aligned}$$

Use phasors.
The phasors corresponding to each component signal are given in Table 1. The complex number equivalent to each phasor is given as well. Adding all complex numbers gives

$$\mathbf{V} = 10(2\sqrt{3} - 1) - j20$$

which corresponds to the phasor

$$\mathbf{V} = \sqrt{17 - 4\sqrt{3}} \angle -\arctan\frac{2}{2\sqrt{3} - 1}$$

which can be transformed into the time signal

$$v(t) = \sqrt{17 - 4\sqrt{3}} \cos(\omega t + -\arctan\frac{2}{2\sqrt{3} - 1})$$

7. P5.33
The frequency is $\omega = 2000\pi\frac{1}{s}$. The complex impedance Z_L of the inductor is

$$Z_L = j\omega L = j2000\pi\frac{1}{s}0.1H = j200\pi\Omega$$

Note that $\omega = 2000\pi$ is the reference frequency. That phasor corresponding to $v_L(t) = 10\cos\omega t)$ is

$$\mathbf{V_L} = 10V\angle 0$$

The phasor corresponding to the current $i_L(t)$ through the inductor is

$$\begin{aligned}
\mathbf{I_L} &= \frac{1}{Z_L}\mathbf{V_L} \\
&= (\frac{1}{200\pi}\angle - \frac{\pi}{2})(10\angle 0)A \\
&= \frac{1}{20\pi}\angle - \frac{\pi}{2}A
\end{aligned}$$

This corresponds to the time function

$$i_L(t) = \frac{1}{20\pi}\cos{(2000\pi\frac{t}{s} - \frac{\pi}{2})}A = \frac{1}{20\pi}\sin{(2000\pi\frac{t}{s})}A$$

The phasor plot is depicted in Figure 3. In Figure 4, the signals are plotted versus time. From both plots, it can be inferred that the voltage is lagging by $\frac{\pi}{2}$.

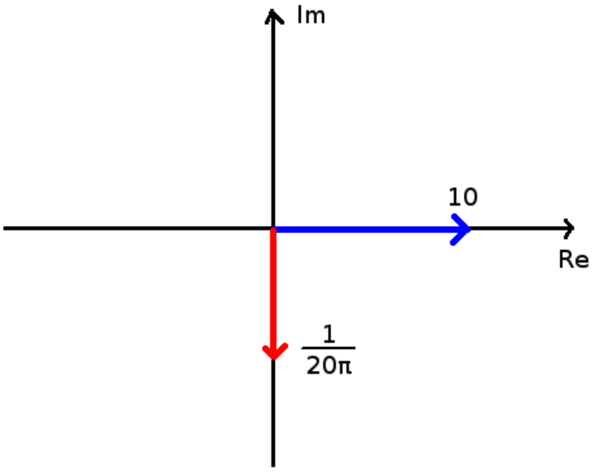

Figure 3: Phasor Diagram

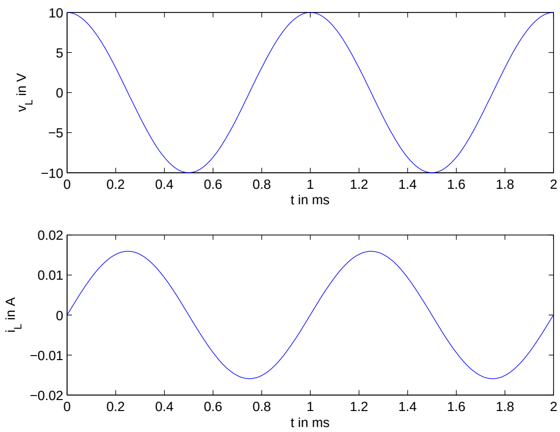

Figure 4: Time Diagram

8. Find the complex impedance in polar form of the network shown in Figure 5 for $\omega = 1000\frac{1}{s}$, $\omega = 2000\frac{1}{s}$, and $\omega = 4000\frac{1}{s}$.
 We have for the impedance

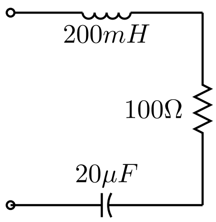

Figure 5: Circuit 1

$$\begin{aligned} Z &= R + j\omega L + \frac{1}{j\omega C} \\ &= 100 + j\omega 0.2 + -\frac{j5 * 10^4}{\omega}\Omega \end{aligned} \tag{3}$$

For $\omega = 500$, this is

$$Z = 100 + j100 - j100\Omega = 100\Omega$$

For $\omega = 1000$, we have

$$Z = 100 + j200 - j50\Omega = 100 + j150\Omega \approx 180.28\angle 56.21^\circ$$

And for $\omega = 2000$, we have

$$Z = 100 + j400 - j25\Omega = 100 + j375\Omega \approx 388.10\angle 75.07^\circ$$

9. P5.47
The frequency of the driving source is $\omega = 100$. Replacing the resistance and the inductance by the corresponding impedances yields the circuit which is drawn in Figure 6 where the current and voltage variables have been replaced by the corresponding phasors.
From KVL, we get

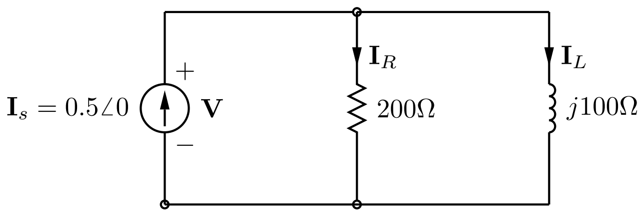

Figure 6: Circuit 3

$$\mathbf{V} = 200\mathbf{I}_R = j100\mathbf{I}_L$$

and from KCL we get

$$\mathbf{I}_s = 0.5\angle 0 = \mathbf{I}_R + \mathbf{I}_L$$

Substituting gives

$$\mathbf{V}\left(\frac{1}{200} + \frac{1}{j100}\right) = 0.5$$

where we have replaced $0.5\angle 0$ by the corresponding complex number.
We can now solve for $\mathbf{V}$:

$$\mathbf{V} = \frac{100j100}{200 + j100} = \frac{j100}{2 + j} = 20 + j40$$

This gives

$$\begin{aligned} \mathbf{I}_R &= \frac{\mathbf{V}}{200} = 0.1 + j0.2 \\ \mathbf{I}_L &= \frac{\mathbf{V}}{j100} = 0.4 - j0.2 \end{aligned}$$

The phasor diagram is depicted in Figure 7. $\mathbf{I}_s$ lags $\mathbf{V}$ by arctan(2). Notice that $\mathbf{I}_R$ is aligned with $\mathbf{V}$ and $\mathbf{I}_L$ lags $\mathbf{V}$ by $\frac{\pi}{2}$. Also $\mathbf{I}_R + \mathbf{I}_L = \mathbf{I}_s$, as required by KCL.

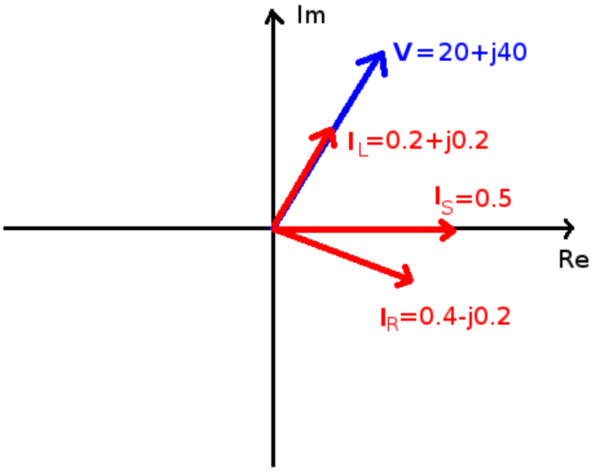

Figure 7: Phasor Diagram

10. P5.50

The frequency of the driving source is $\omega 10^4 \frac{1}{s}$. Using the corresponding impedances of the circuit elements and replacing all currents and cvoltages by their corresponding phasors we may redraw the circuit as in Figure 8.

We have imediately

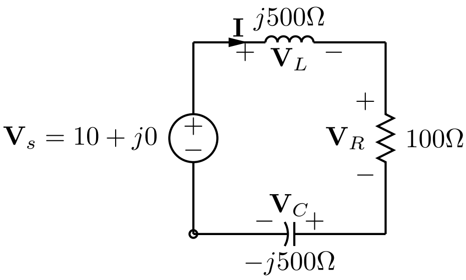

Figure 8: Circuit 4

$$
\begin{aligned}
\mathbf{V}_L &= j500\mathbf{I} \\
\mathbf{V}_R &= 100\mathbf{I} \\
\mathbf{V}_C &= -j500\mathbf{I}
\end{aligned}
$$

Using KVL around the loop, we get

$$10 + j0 = \mathbf{V}_S = 100\mathbf{I}$$

This gives $\mathbf{I} = 0.1 + j0$, and we calculate

$$
\begin{aligned}
\mathbf{V}_L &= j50\mathbf{I} \\
\mathbf{V}_R &= 10\mathbf{I} \\
\mathbf{V}_C &= -j50\mathbf{I}
\end{aligned}
$$

The phasor $\mathbf{V}_L = j50$ corresponds to the time function

$$v_L(t) = -50V \sin(10^4 \frac{t}{s})$$

This has the peak value $50V$, which is much larger than the peak value $10V$ of the driving voltage source. This is evidence of the fact that there is a lof of reactive power sloshing around in the circuit.

The voltage across the inductor leads the current through it by $\frac{\pi}{2}$, so the power angle is $\frac{\pi}{2}$ (see eqn. (5.63)) and the power factor is zero (see eqn. (5.61)). The power delivered to the inductor is

$$p(t) = \frac{5}{2}\sin(2*10^4\frac{t}{s})W$$

(see page 225) and is purely reactive with peak instantaneous value

$$\frac{5}{2} = \frac{50}{\sqrt{2}}\frac{0.1}{\sqrt{2}}\sin(\frac{\pi}{2})$$

where we have written it in the form of eqn. (5.63).
Likewise, the voltage across the capacitor lags the current through it by $\frac{\pi}{2}$, so the corresponding power angle is $-\frac{\pi}{2}$, the power factor is again zero, and the power delivered to the capacitor is

$$p(t) = -\frac{5}{2}\sin(2*10^4\frac{t}{s})W$$

(see page 226) which is purely reactive. What is going on is that the inductor and the capacitor (the reactive elements in the circuit) are exchanging energy between each other.
The power delivered to the resistor is $1W$ and this is the true power dissipated in the circuit.