EE40
Lecture 12
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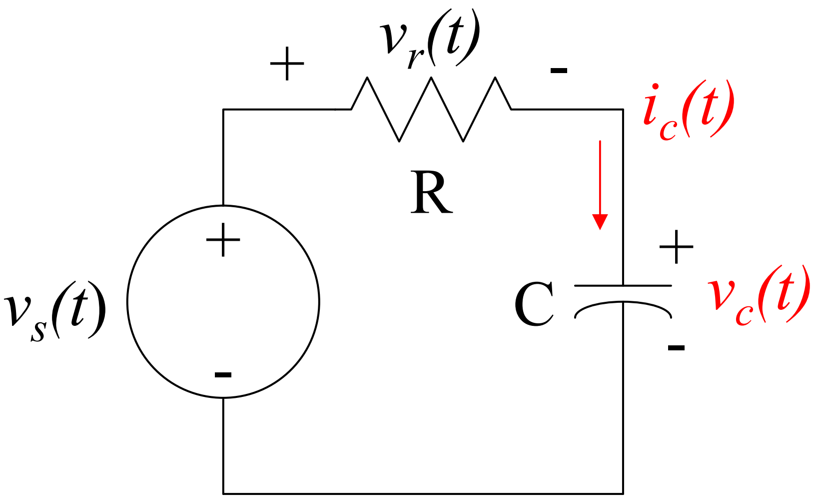
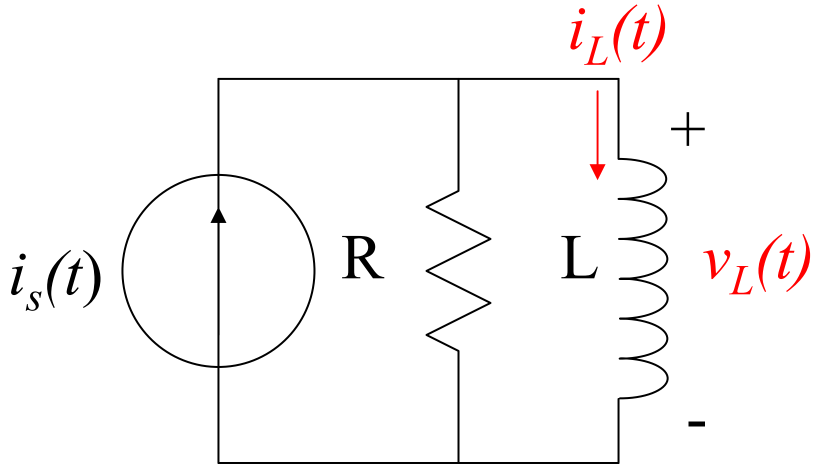
2/20/08

Reading: Chap. 4: first order circuits

First Order Circuits





KVL around the loop:

KCL at the node:

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)$$

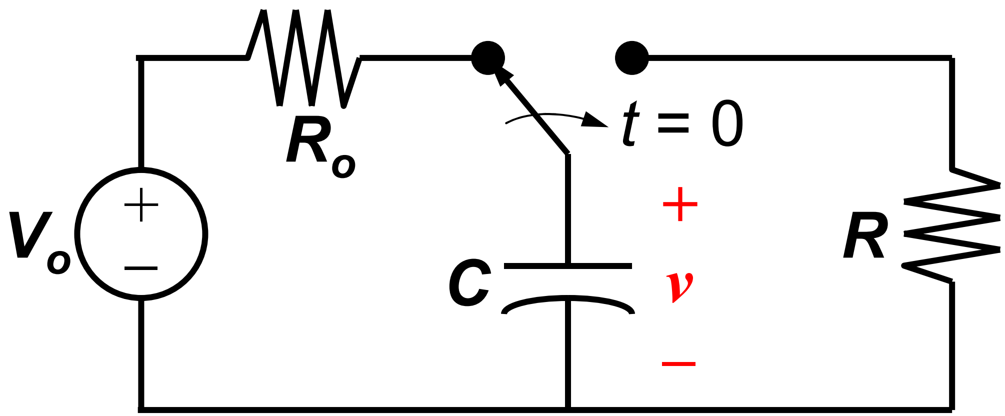
$$\frac{L}{R}\frac{di_L(t)}{dt}+i_L(t)=i_s(t)$$

Response of a Circuit

- ***Transient response*** of a circuit is the portion of the total response that dies to zero as time goes to infinity.
- ***Steady-state response*** is what remains of the total response after the transient response is removed.
- ***Natural response*** of a circuit is the portion of the response due to the initial conditions. This is generally part of the transient response, except in cases where there is some energy in the initial conditions that cannot dissipate through resistors (e.g. a pure LC oscillator).
- ***Forced response*** is the portion of the response due to the forcing function (the right hand side of the differential equation that comes from the sources). In general this will also have a transient part.
- We will later be mostly interested in the steady state response of circuits to a ***DC*** forcing function or more generally a ***sinusoidal*** forcing function
- Note: some of this terminology is at variance with your book. Your book's definition of ``forced response". for instance, is inconsistent.

Natural Response of an RC Circuit

- Consider the following circuit, for which the switch is closed for $t < 0$, and then opened at $t = 0$:



Notation:

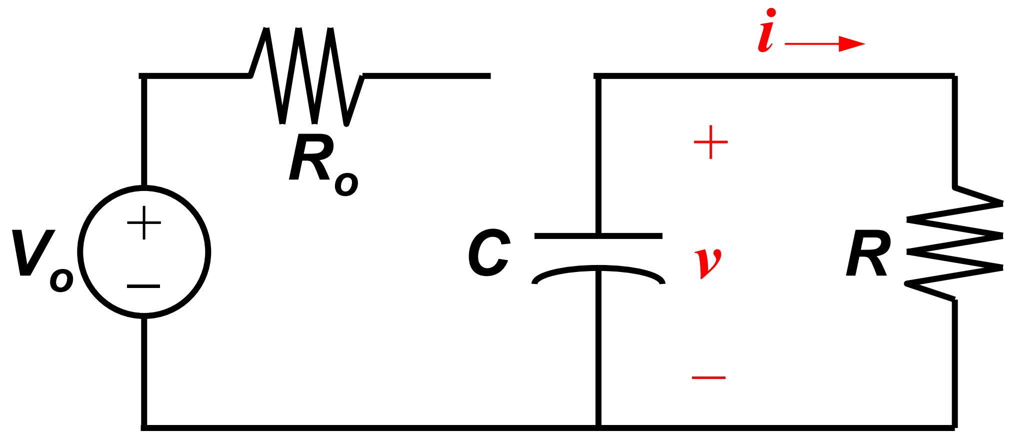
0^- is used to denote the time just prior to switching

0^+ is used to denote the time immediately after switching

- The voltage on the capacitor at $t = 0^-$ is V_o

Solving for the Voltage ($t \geq 0$)

- For t > 0, the circuit reduces to

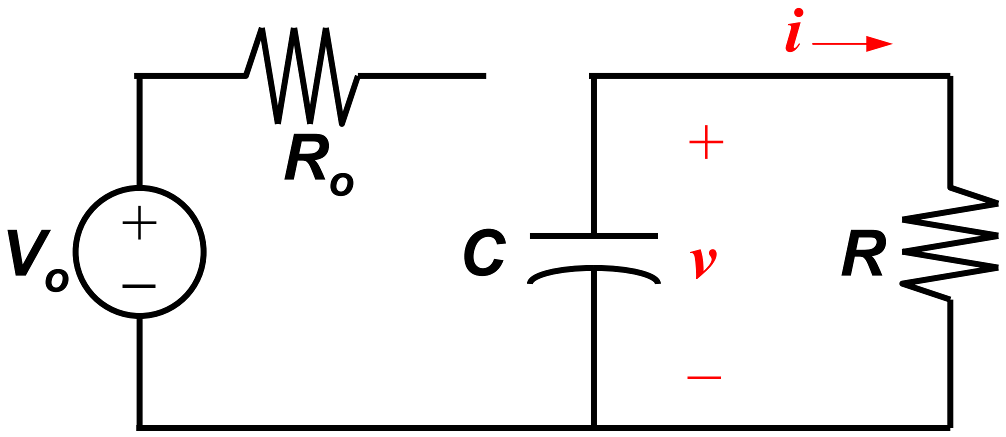


- Applying KCL to the RC circuit:

- Solution:

$$v(t) = \mathrm{V_o}\, e^{-t/RC}$$

Solving for the Current (t > 0)



$$v(t) = V_o e^{-t/RC}$$

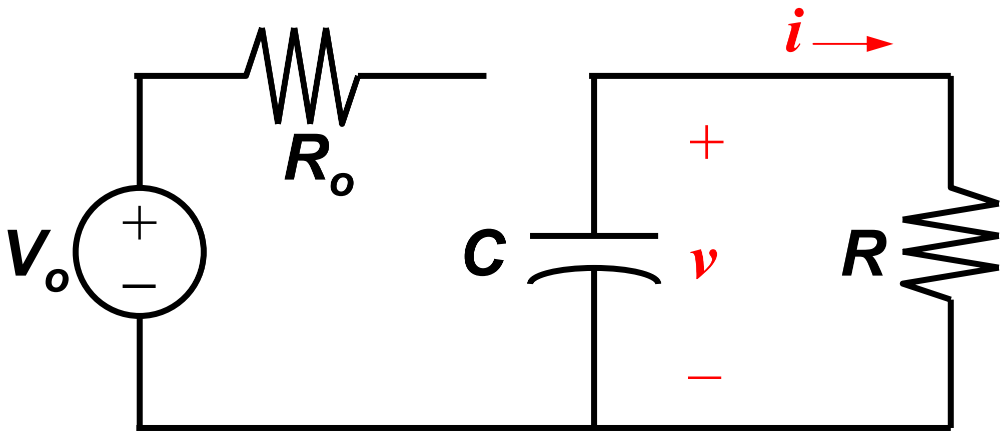
- Note that the current changes abruptly:

$$i(0^-) = 0$$

$$\text{for } \; t > 0, \; i(t) = \frac{v}{R} = \frac{V_o}{R} e^{-t/RC}$$

$$\Rightarrow \; i(0^+) = \frac{V_o}{R}$$

Solving for Power and Energy Delivered (t > 0)



$$v(t) = V_o e^{-t/RC}$$

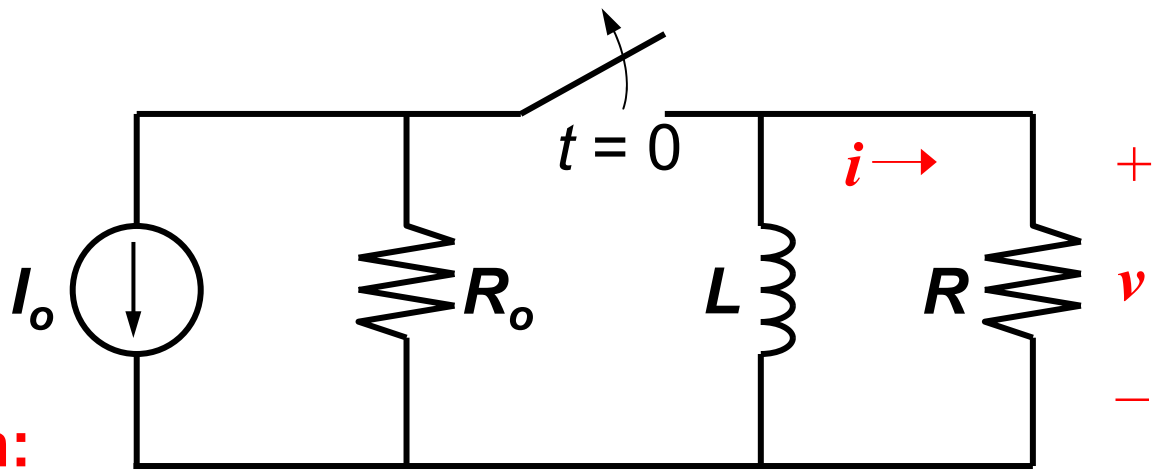
$$p = \frac{v^2}{R} = \frac{V_o^2}{R} e^{-2t/RC}$$

$$w = \int_0^t p(x)dx = \int_0^t \frac{V_o^2}{R} e^{-2x/RC} dx$$

$$= \frac{1}{2} C V_o^2 \left(1 - e^{-2t/RC}\right)$$

Natural Response of an RL Circuit

- Consider the following circuit, for which the switch is closed for $t < 0$, and then opened at $t = 0$:



Notation:

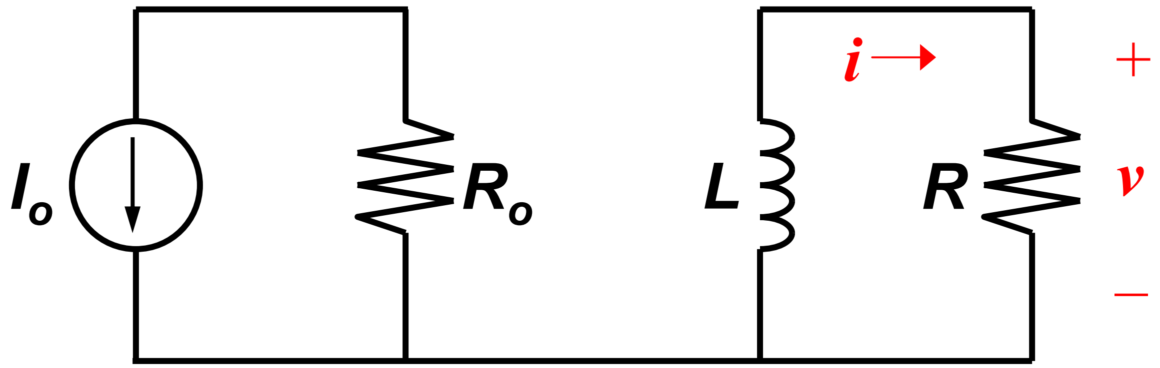
0^- is used to denote the time just prior to switching

0^+ is used to denote the time immediately after switching

- t<0 the entire system is at steady-state; and the inductor is → like short circuit
- The current flowing in the inductor at $t = 0^-$ is I_o and V across is 0.

Solving for the Current ($t \geq 0$)

- For t > 0, the circuit reduces to

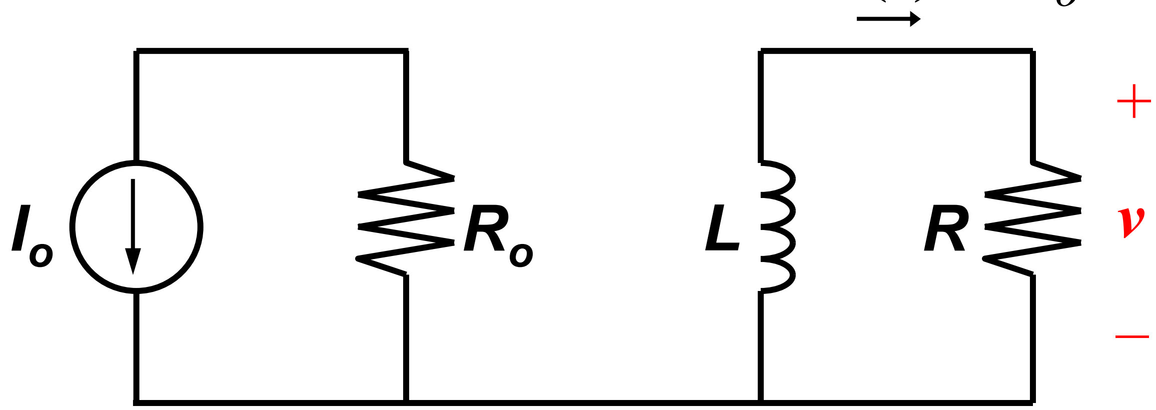


- Applying KVL to the LR circuit:
- $v(t) = i(t)R$
- At $t=0^+$, $i = I_{0,}$
- At arbitrary t>0, $i=i(t)$ and $v(t) = -L\dfrac{di(t)}{dt}$

- Solution: $i(t) = i(0)e^{-(R/L)t} = I_0 e^{-(R/L)t}$

Solving for the Voltage ($t > 0$)

$$i(t) = I_o e^{-(R/L)t}$$



- Note that the voltage changes abruptly:

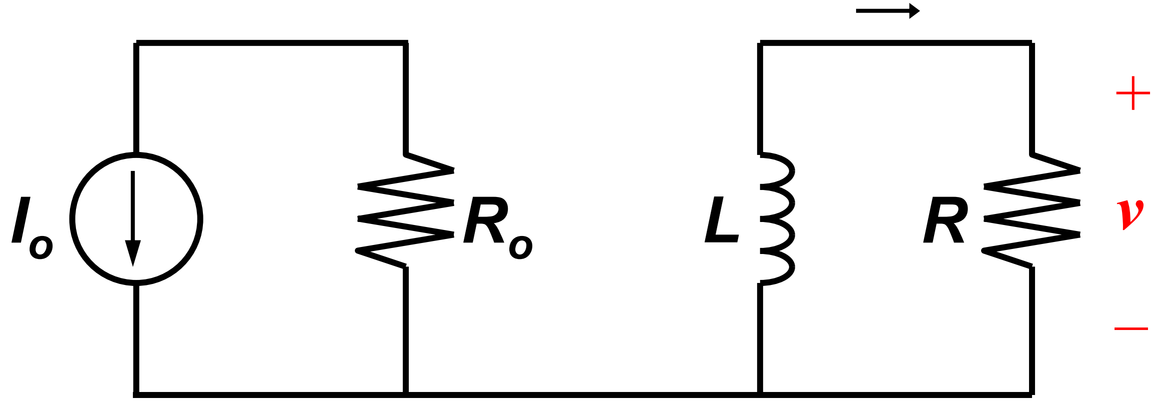
$$v(0^-) = 0$$

$$\text{for } t > 0,\ \ v(t) = iR = I_o R e^{-(R/L)t}$$

$$\Rightarrow\ \ v(0^+) = I_0 R$$

Solving for Power and Energy Delivered (*t* > 0)

$$i(t) = I_o e^{-(R/L)t}$$



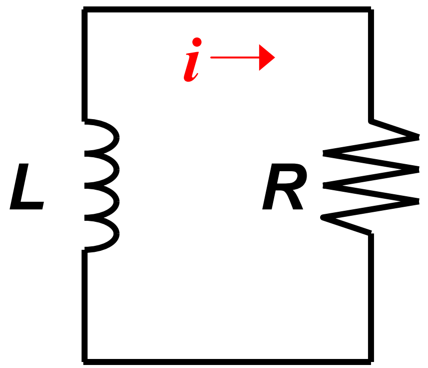
$$p = i^2 R = I_o^2 R e^{-2(R/L)t}$$

$$w = \int_0^t p(x)dx = \int_0^t I_o^2 R e^{-2(R/L)x} dx$$

$$= \frac{1}{2} L I_o^2 \left(1 - e^{-2(R/L)t}\right)$$

Natural Response Summary

RL Circuit



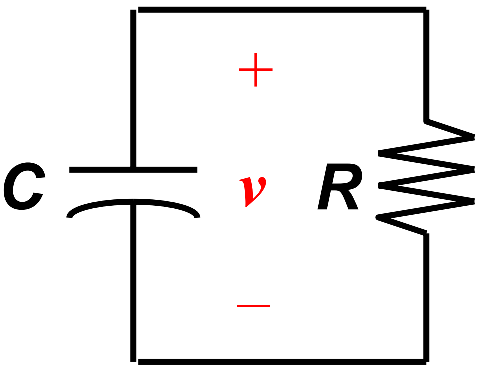
- Inductor current cannot change instantaneously

$$i(0^-) = i(0^+)$$

$$i(t) = i(0)e^{-t/\tau}$$

- time constant $\tau = \dfrac{L}{R}$

RC Circuit



- Capacitor voltage cannot change instantaneously

$$v(0^-) = v(0^+)$$

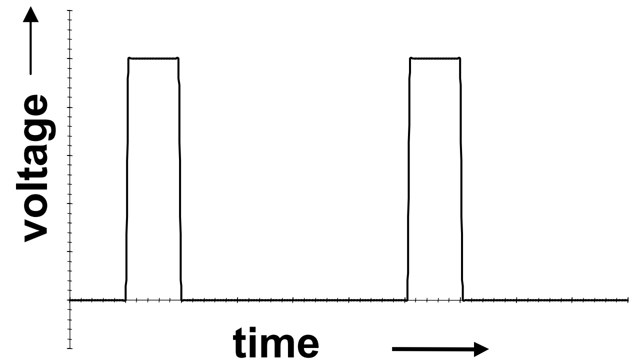
$$v(t) = v(0)e^{-t/\tau}$$

- time constant $\tau = RC$

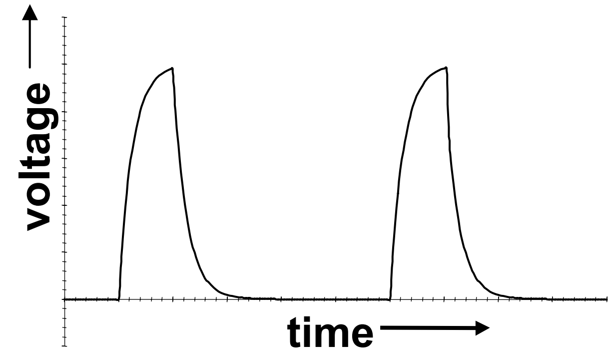
Digital Signals

We compute with pulses.

We send beautiful pulses in:



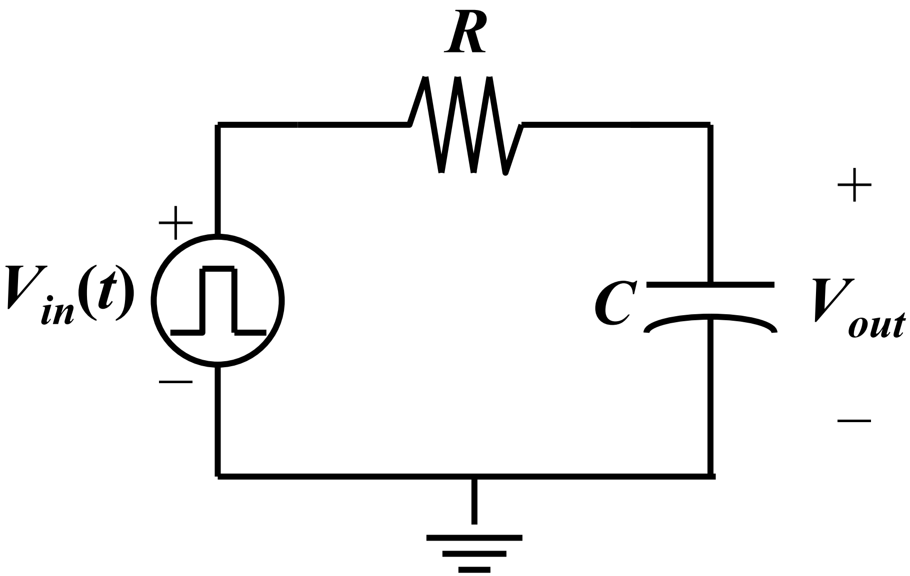
But we receive lousy-looking pulses at the output:



Capacitor charging effects are responsible!

- Every node in a real circuit has capacitance; it's the charging of these capacitances that limits circuit performance (speed)

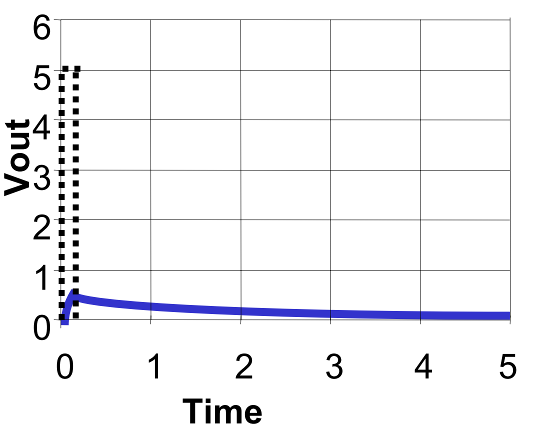
Pulse Distortion



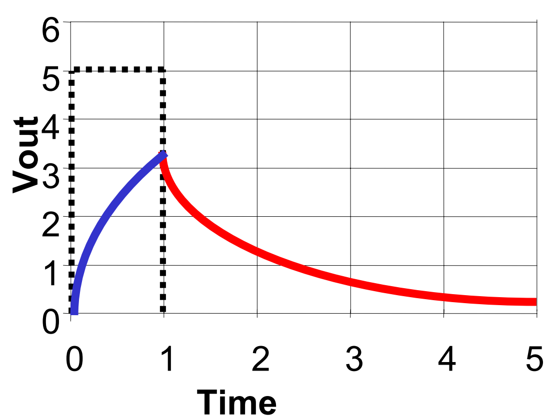
The input voltage pulse width must be large enough; otherwise the output pulse is distorted.

(We need to wait for the output to reach a recognizable logic level, before changing the input again.)

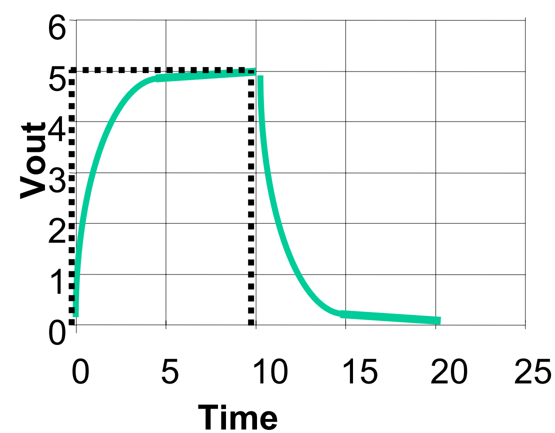
Pulse width = 0.1*RC*



Pulse width = *RC*

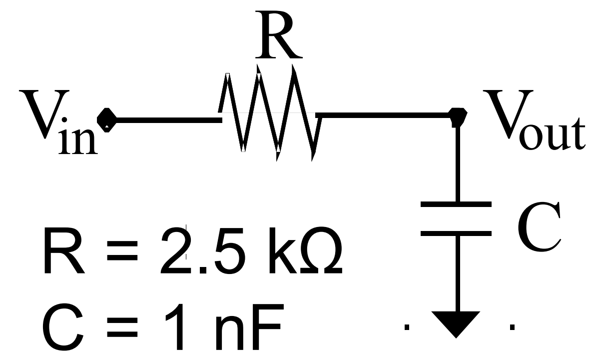


Pulse width = 10*RC*



Example

Suppose a voltage pulse of width 5 μs and height 4 V is applied to the input of this circuit beginning at t = 0:



R = 2.5 kΩ
C = 1 nF

$$\tau = RC = 2.5\ \mu s$$

- First, V_{out} will increase exponentially toward 4 V.
- When V_{in} goes back down, V_{out} will decrease exponentially back down to 0 V.

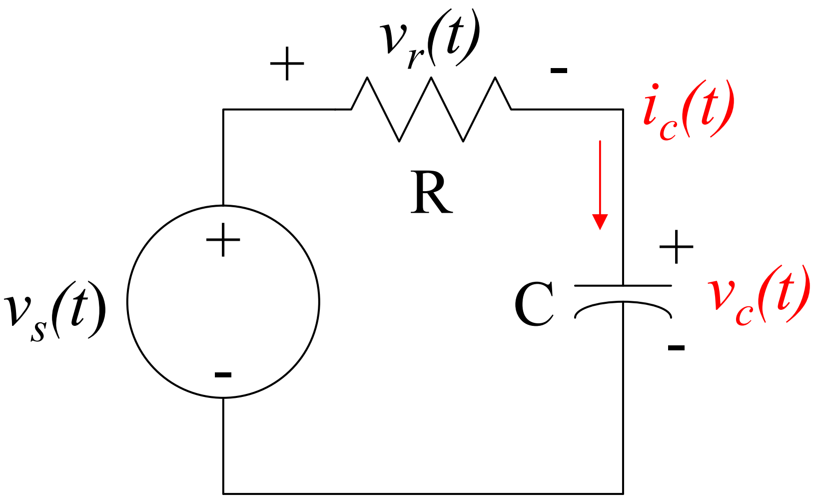
What is the peak value of V_{out}?

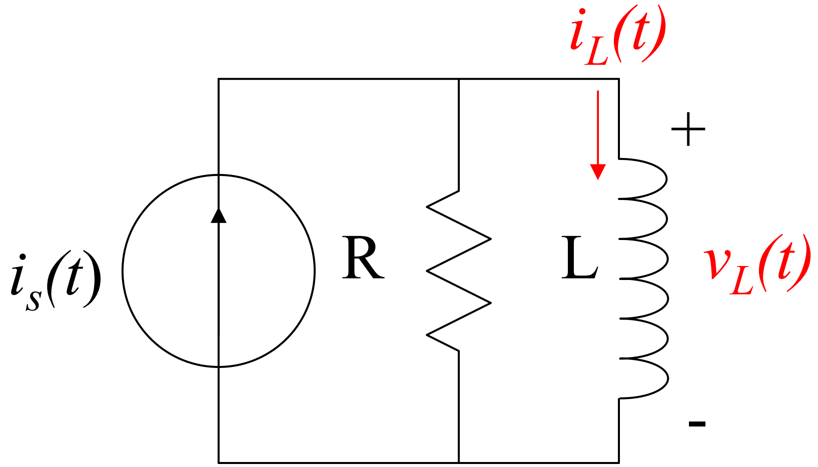
The output increases for 5 μs, or 2 time constants.

→ It reaches $1-e^{-2}$ or 86% of the final value.

0.86 x 4 V = 3.44 V is the peak value

First Order Circuits: Forced Response





KVL around the loop:

$v_r(t) + v_c(t) = v_s(t)$

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

KCL at the node:

$$\frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(x)dx = i_s(t)$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

Complete Solution

- Voltages and currents in a 1st order circuit satisfy a differential equation of the form

$$x(t) + \tau \frac{dx(t)}{dt} = f(t)$$

  - *f(t)* is called the forcing function.
- The complete solution is the sum of any particular solution and the associated complementary solution

$$x(t) = x_p(t) + x_c(t)$$

  - Particular solution is any solution that satisfies the original equation.
  - Complementary solution is a solution of the homogeneous equation (the one with zero forcing function) used to satisfy the initial conditions. In effect it also compensates for the value of the particular solution at 0.

$$x_p(t) + \tau \frac{dx_p(t)}{dt} = f(t)$$

Original equation

$$x_c(t) + \tau \frac{dx_c(t)}{dt} = 0$$

Homogeneous equation

$$x_c(t) = Ke^{-t/\tau}$$

Solution to the homogeneous equation.

The Time Constant

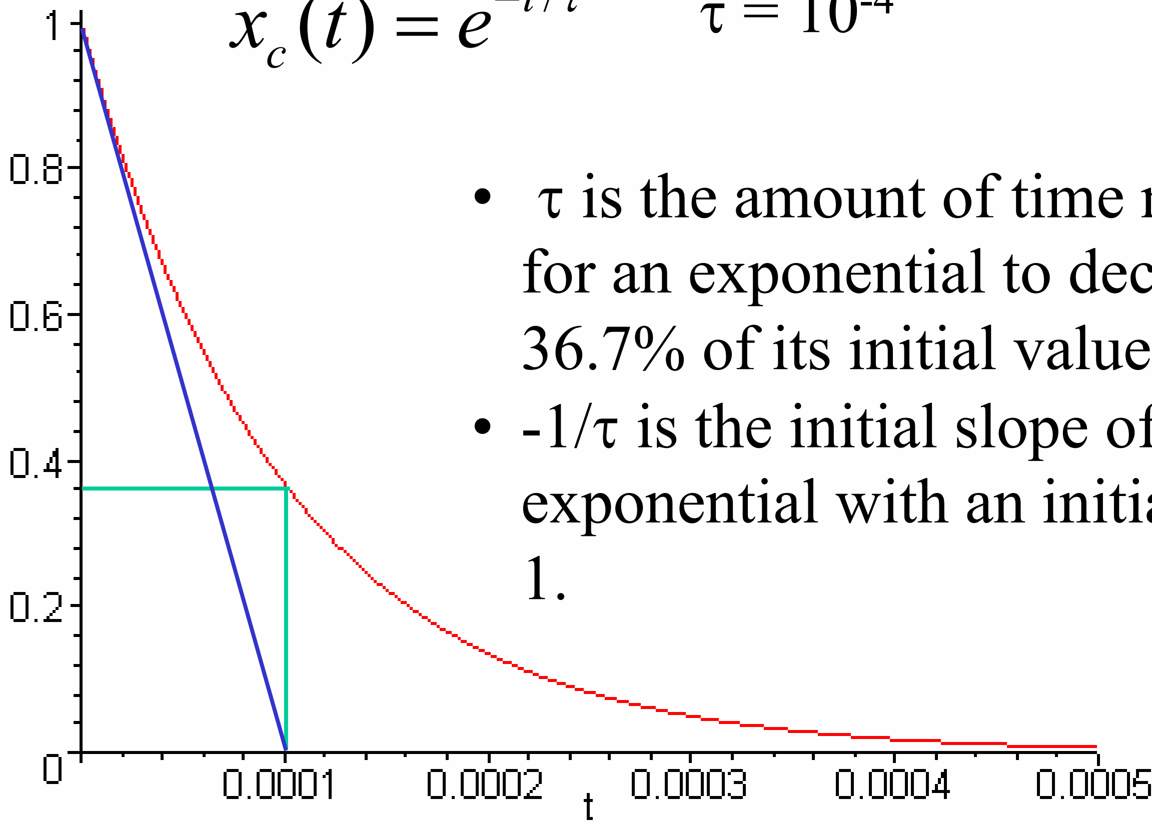
- The complementary solution for any 1st order circuit looks like, for some K,

$$x_c(t) = Ke^{-t/\tau}$$

- For an RC circuit, $\tau = RC$
- For an RL circuit, $\tau = L/R$

What Does $X_c(t)$ Look Like?

$$x_c(t) = e^{-t/\tau} \qquad \tau = 10^{-4}$$



- τ is the amount of time necessary for an exponential to decay to 36.7% of its initial value.
- $-1/\tau$ is the initial slope of an exponential with an initial value of 1.

Particular Solution

- A particular solution $x_p(t)$ is usually found as a weighted sum of $f(t)$ and its first derivative.
- For example, if $f(t)$ is constant (DC forcing function), then $x_p(t)$ can be taken to be a constant.
- If $f(t)$ is sinusoidal, then $x_p(t)$ can be taken to be sinusoidal.
- Note: The notion of ``particular solution" is not unique. Any solution of the homogeneous equation can be added to a particular solution to give another particular solution! The corresponding complementary solution would then adjust accordingly to give the same total response for the given initial conditions.

Particular Solution: F(t) Constant

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F$$

Guess a solution

$$x_P(t) = A + Bt$$

$$(A + Bt) + \tau \frac{d(A + Bt)}{dt} = F$$

Equation holds for all time and thus each coefficient is individually zero

$$(A + Bt) + \tau B = F$$

$$(A + \tau B - F) + (B)t = 0$$

$$(B) = 0$$

$$(A + \tau B - F) = 0$$

$$B = 0$$

$$A = F$$

The Particular Solution: F(t) Sinusoid

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

Guess a solution $\quad x_P(t) = A\sin(wt) + B\cos(wt)$

$$(A\sin(wt) + B\cos(wt)) + \tau \frac{d(A\sin(wt) + B\cos(wt))}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$(A - \tau\omega B - F_A)\sin(\omega t) + (B + \tau\omega A - F_B)\cos(\omega t) = 0$$

$$(A - \tau\omega B - F_A) = 0 \qquad (B + \tau\omega A - F_B) = 0$$

Equation holds for all time and \
thus each coefficient is
individually zero

$$A = \frac{F_A + \tau\omega F_B}{(\tau\omega)^2 + 1} \qquad B = -\frac{\tau\omega F_A - F_B}{(\tau\omega)^2 + 1}$$

$$x_P(t) = \frac{1}{\sqrt{(\tau\omega)^2 + 1}} \left[\frac{\tau\omega}{\sqrt{(\tau\omega)^2 + 1}} \sin(\omega t) + \frac{1}{\sqrt{(\tau\omega)^2 + 1}} \cos(\omega t) \right]$$

$$= \frac{1}{\sqrt{(\tau\omega)^2 + 1}} \cos(\omega t - \theta); \quad where\ \theta = \tan^{-1}(\tau\omega)$$

The Particular Solution: F(t) Exp.

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_1 e^{-\alpha t} + F_2$$

Guess a solution

$$x_P(t) = A + Be^{-\alpha t}$$

$$(A + Be^{-\alpha t}) + \tau \frac{d(A + Be^{-\alpha t})}{dt} = F_1 e^{-\alpha t} + F_2$$

Equation holds for all time and thus each coefficient is individually zero

$$(A + Be^{-\alpha t}) - \alpha\tau Be^{-\alpha t} = F_1 e^{-\alpha t} + F_2$$

$$(A - F_2) + (B - \alpha\tau - F_1)e^{-\alpha t} = 0$$

$$(A - F_2) = 0$$

$$(B - \alpha\tau - F_1) = 0$$

$$A = F_2$$

$$B = \alpha\tau + F1$$

The Total Solution: F(t) Sinusoid

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$x_P(t) = A\sin(wt) + B\cos(wt) \qquad A = \frac{F_A + \tau\omega F_B}{(\tau\omega)^2 + 1} \qquad B = -\frac{\tau\omega F_A - F_B}{(\tau\omega)^2 + 1}$$

$$x_C(t) = Ke^{-t/\tau}$$

$$x_T(t) = A\sin(wt) + B\cos(wt) + Ke^{-t/\tau}$$

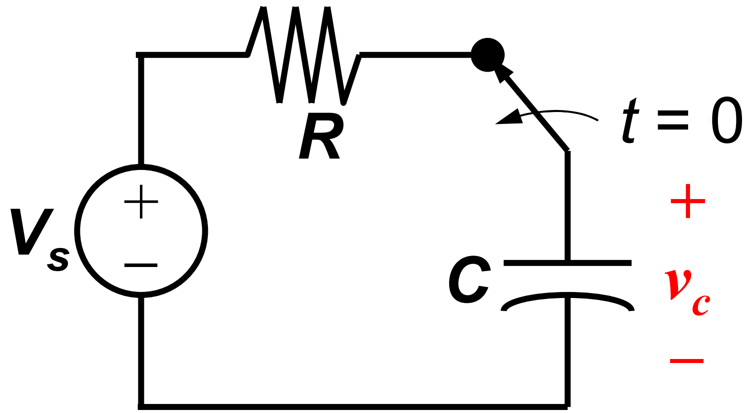
Only K is unknown and is determined by the initial condition at t =0

Example: $x_T(t=0) = V_C(t=0)$

$$x_T(0) = A\sin(0) + B\cos(0) + Ke^{-0/\tau} = V_C(t=0)$$

$$x_T(0) = B + K = V_C(t=0) \qquad K = V_C(t=0) - B$$

Example



- Given $v_c(0^-)=1$, $V_s=2\,cos(\omega t)$, $\omega=200$.
- *Find i(t)*, $v_c(t)=?$