EE40
Lecture 13
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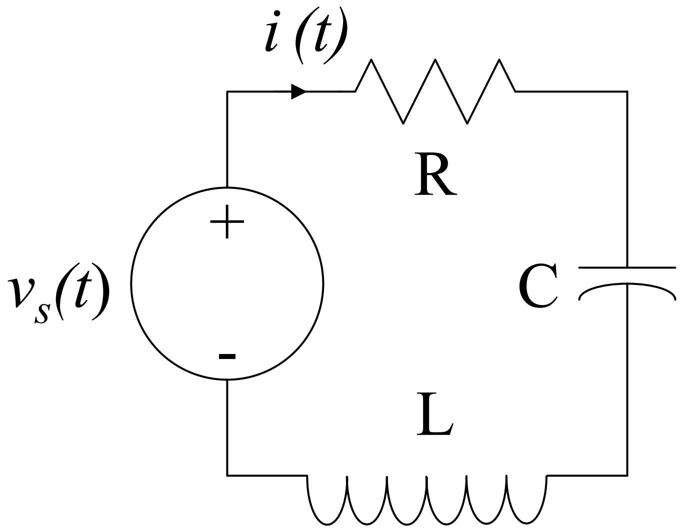
Reading: Chap. 4: second order circuits

2nd Order Circuits

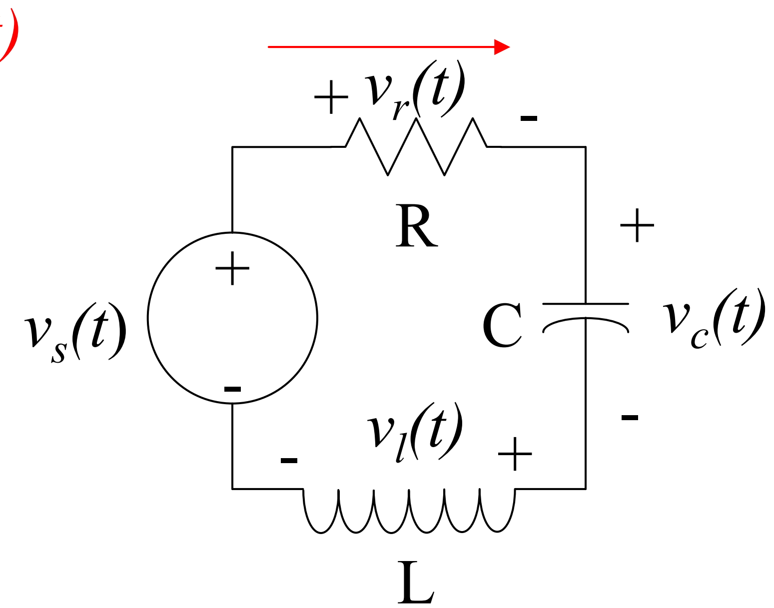
- We consider simple circuits with a single capacitor and a single inductor.
- Any voltage or current in such a circuit results from the solution to a 2nd order differential equation. Hence such circuits are called second order circuits.

A 2nd Order RLC Circuit



- Application: Filters
  - A bandpass filter such as the IF amp for the AM radio.
  - A lowpass filter with a sharper cutoff than can be obtained with an RC circuit.

The Differential Equation



KVL around the loop:

$$v_r(t) + v_c(t) + v_l(t) = v_s(t)$$

$$Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(x)dx + L\frac{di(t)}{dt} = v_s(t)$$

$$\frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) + \frac{d^2 i(t)}{dt^2} = \frac{1}{L}\frac{dv_s(t)}{dt}$$

The Differential Equation

The voltage and current in a second order circuit is the solution to a differential equation of the following form:

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)$$

$$x(t) = x_p(t) + x_c(t)$$

$X_p(t)$ is any particular solution and $X_c(t)$ is the complementary solution (solution of the homogeneous equation chosen so that the total solution matches the initial conditions).

The Particular Solution

- A particular solution $x_p(t)$ can usually be chosen as a weighted sum of $f(t)$ and its first and second derivatives.
- If $f(t)$ is constant, then $x_p(t)$ can be chosen to be constant. Its value is determined by the equation.
- If $f(t)$ is sinusoidal, then $x_p(t)$ can be chosen to be sinusoidal with the same frequency. The magnitude and phase are determined by the equation.

The Complementary Solution

To find the general form of the solution of the homogeneous equation we may start with trying the following form:

$$x_c(t) = Ke^{st}$$

s must satisfy an algebraic equation determined by the coefficients of the differential equation:

$$\frac{d^2Ke^{st}}{dt^2} + 2\alpha\frac{dKe^{st}}{dt} + \omega_0^2 Ke^{st} = 0$$

$$s^2Ke^{st} + 2\alpha sKe^{st} + \omega_0^2 Ke^{st} = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Characteristic Equation

- This algebraic equation is called the characteristic equation, and we must find its roots

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

$$\alpha = \zeta\omega_0$$

- The characteristic equation has two (not necessarily distinct) roots. When the roots are distinct the general form of the solution of the homogeneous equation is the possibly complex function:

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$

Damping Ratio and Natural Frequency

$$\zeta = \frac{\alpha}{\omega_0}$$

damping ratio

$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}$$

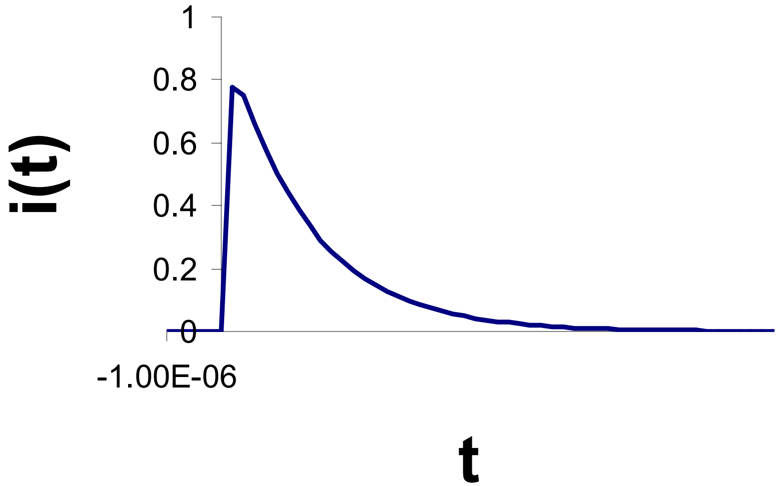
$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$

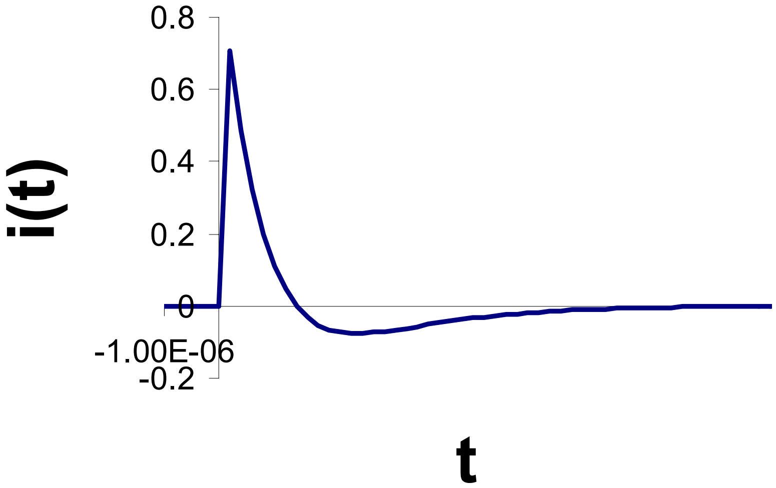
- The damping ratio determines what type of solution we will get:
  - Exponentially decreasing (ζ >1)
  - Exponentially decreasing sinusoid (ζ < 1)
- The undamped natural frequency is ω_0

Overdamped : Real Unequal Roots

- If $\zeta > 1$, s_1 and s_2 are real and not equal.

$$i(t) = K_1 e^{\left(-\varsigma\omega_0 + \omega_0\sqrt{\varsigma^2 - 1}\right)t} + K_2 e^{\left(-\varsigma\omega_0 - \omega_0\sqrt{\varsigma^2 - 1}\right)t}$$



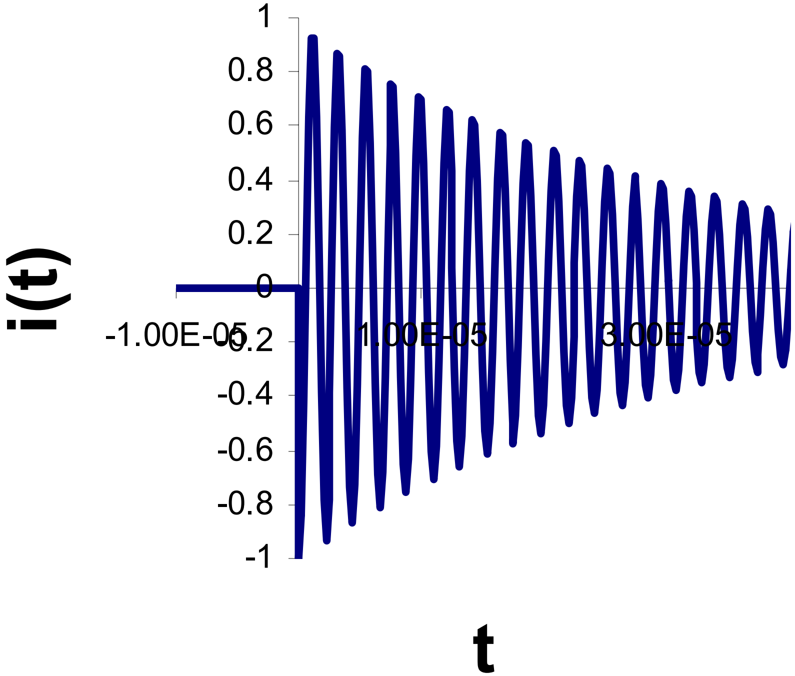


Underdamped: Complex Roots

- If $\zeta < 1$, s_1 and s_2 are complex.
- Define the following constants:

$$\alpha = \zeta\omega_0 \qquad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

$$x_c(t) = e^{-\alpha t}\left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)$$



Critically damped: Real Equal Roots

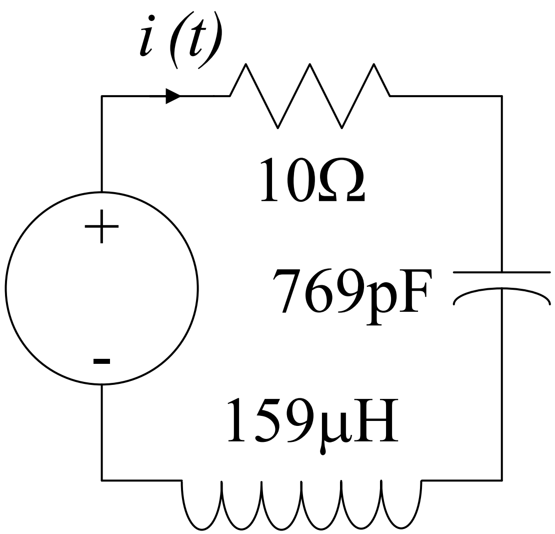
- If $\zeta = 1$, s_1 and s_2 are real and equal.

$$x_c(t) = K_1 e^{-\varsigma\omega_0 t} + K_2 t e^{-\varsigma\omega_0 t}$$

Note: The degeneracy of the roots results in the extra factor of 't'

Example

For the example, what are ζ and ω_0?



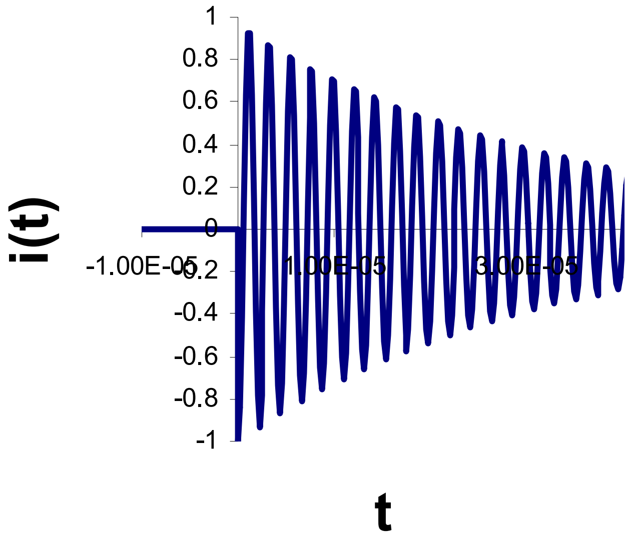
$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

$$\frac{d^2 x_c(t)}{dt^2} + 2\zeta\omega_0 \frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$

$$\omega_0^2 = \frac{1}{LC} \;,\; 2\zeta\omega_0 = \frac{R}{L} \;,\; \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

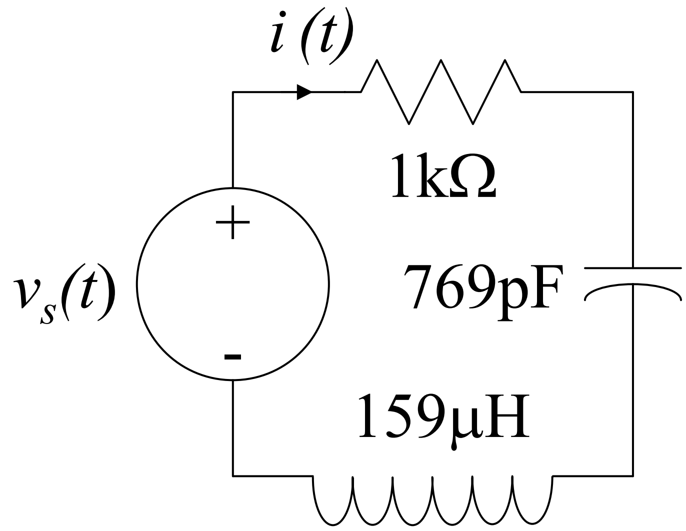
Example

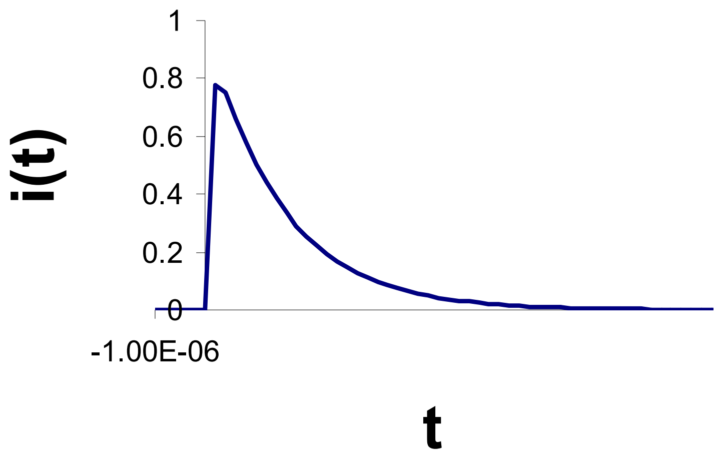
- $\zeta = 0.011$
- $\omega_0 = 2\pi 455000$
- Is this system over damped, under damped, or critically damped?
- What will the current look like?



Slightly Different Example

- Increase the resistor to 1kΩ
- What are ζ and ω_0?





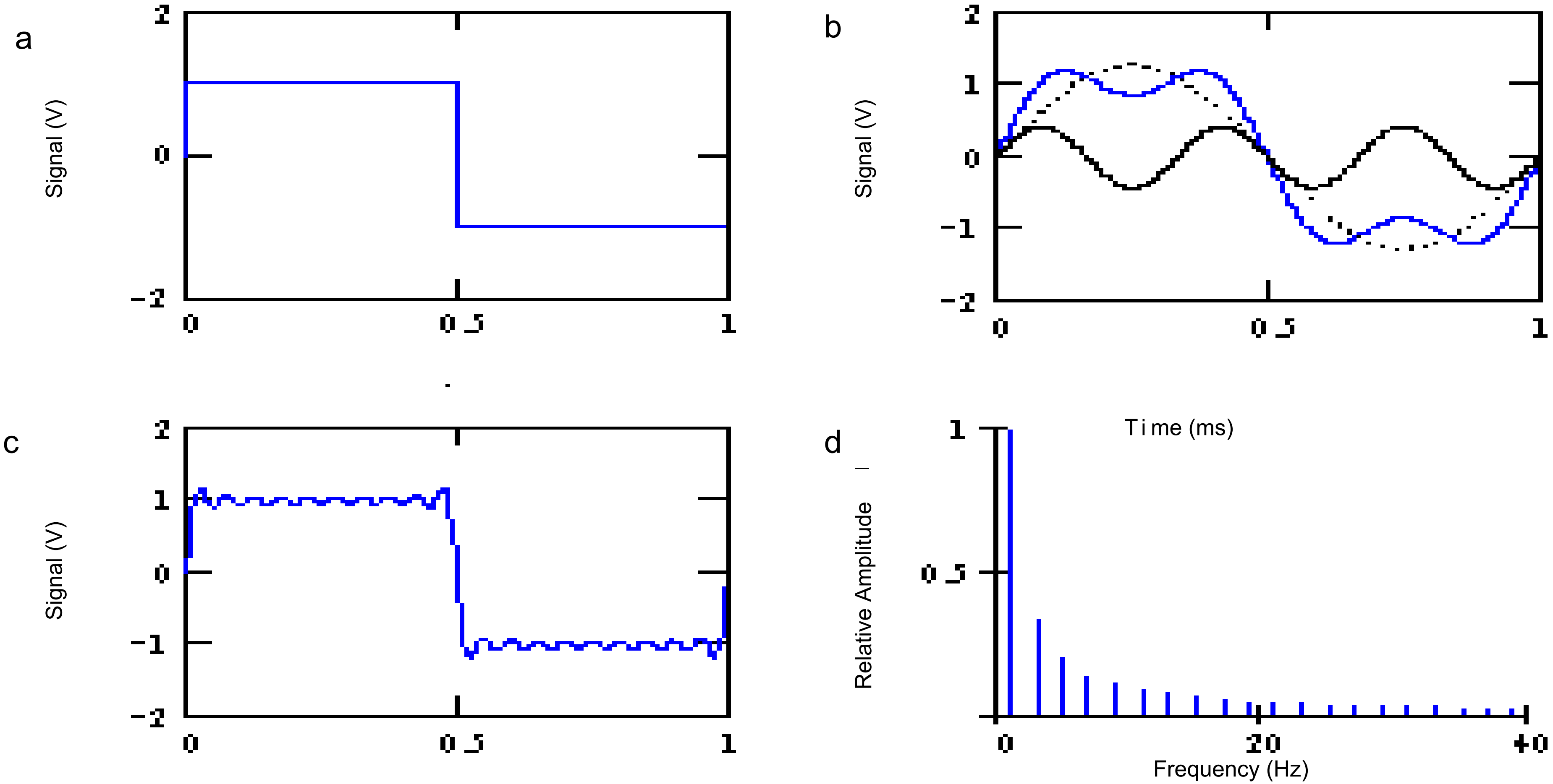
$$\zeta = 2.2$$
$$\omega_0 = 2\pi 455000$$

Why is Single-Frequency Excitation Important?

- Some circuits are driven by a single-frequency sinusoidal source.
- You can express any periodic signal as a sum of single-frequency sinusoids – so you can analyze the response of the (linear, time-invariant) circuit to each individual frequency component and then sum the responses to get the total response.

- This is known as the Fourier Transform technique and is tremendously important in all kinds of engineering disciplines!

Representing a Square Wave as a Sum of Sinusoids



(a)Square wave with 1-second period. (b) Fundamental component (dotted) with 1-second period, third-harmonic (solid black) with1/3-second period, and their sum (blue). (c) Sum of first ten components. (d) Spectrum with 20 terms.

Steady-State Sinusoidal Analysis

- Also known as AC steady-state
- Any steady state voltage or current in a linear circuit with a sinusoidal source is a sinusoid.
  - This is a consequence of the nature of particular solutions for sinusoidal forcing functions.
- All AC steady state voltages and currents have the same frequency as the source.
- In order to find a steady state voltage or current, all we need to know is its magnitude and its phase relative to the source
  - We already know its frequency.