EE40
Lecture 14
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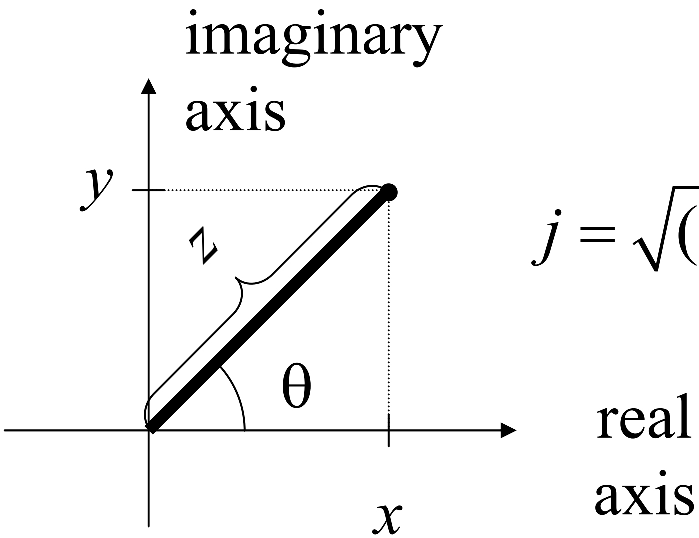
2/27/08

Reading: Chap. 5: phasors

Complex Numbers (1)



$$j = \sqrt{(-1)}$$

- *x* is the real part
- *y* is the imaginary part
- *z* is the magnitude
- θ is the phase

$$x = z\cos\theta \qquad y = z\sin\theta$$

$$z = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

- Rectangular Coordinates

  Z = *x* + *jy*

- Polar Coordinates:

  Z = *z* ∠ θ

- Exponential Form:

$$\mathbf{Z} = |\mathbf{Z}| e^{j\theta} = z e^{j\theta}$$

$$\mathbf{Z} = z(\cos\theta + j\sin\theta)$$

$$1 = 1e^{j0} = 1\angle 0^\circ$$

$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90^\circ$$

Complex Numbers (2)

Euler's Identities

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Exponential Form of a complex number

$$\mathbf{Z} = \left|\mathbf{Z}\right| e^{j\theta} = z e^{j\theta} = z\angle\theta$$

Arithmetic With Complex Numbers

- To compute phasor voltages and currents, we need to be able to perform computations with complex numbers.
  - Addition
  - Subtraction
  - Multiplication
  - Division

Addition

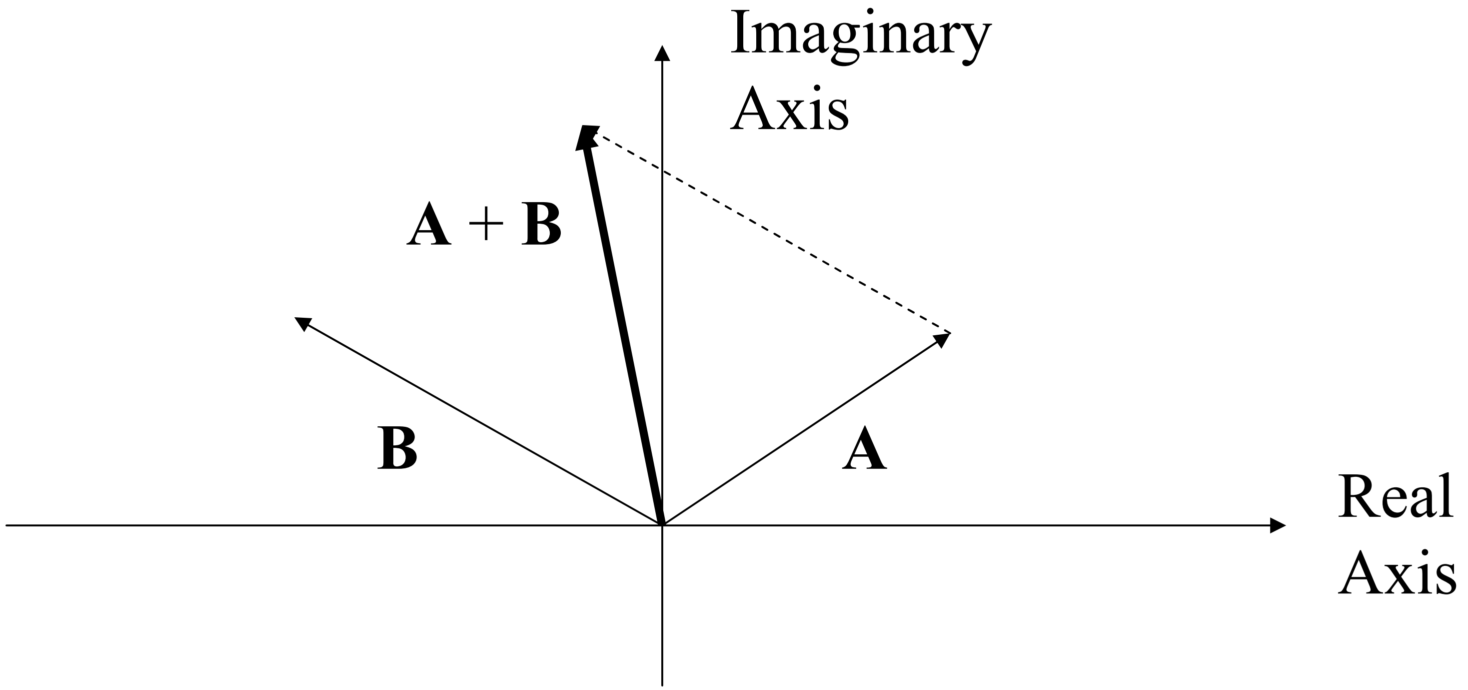
- Addition is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} + \mathbf{B} = (x + z) + j(y + w)$$

Addition



Subtraction

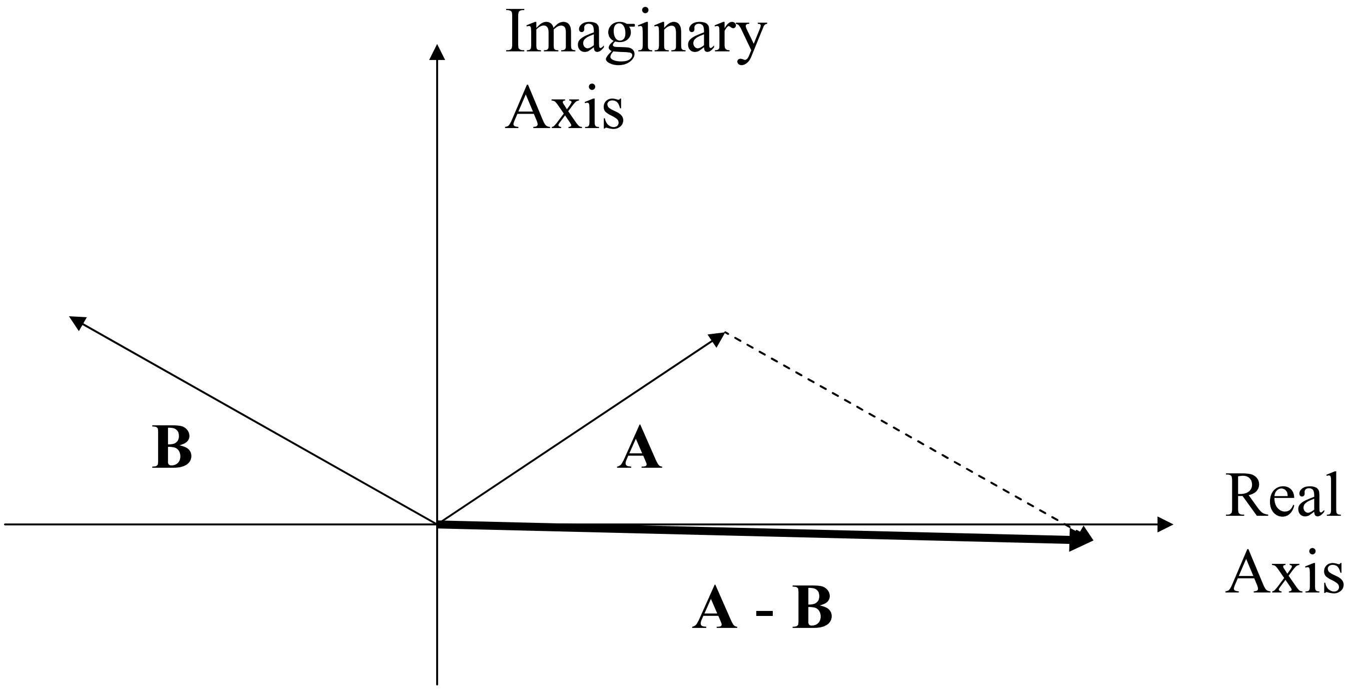
- Subtraction is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} - \mathbf{B} = (x - z) + j(y - w)$$

Subtraction



Multiplication

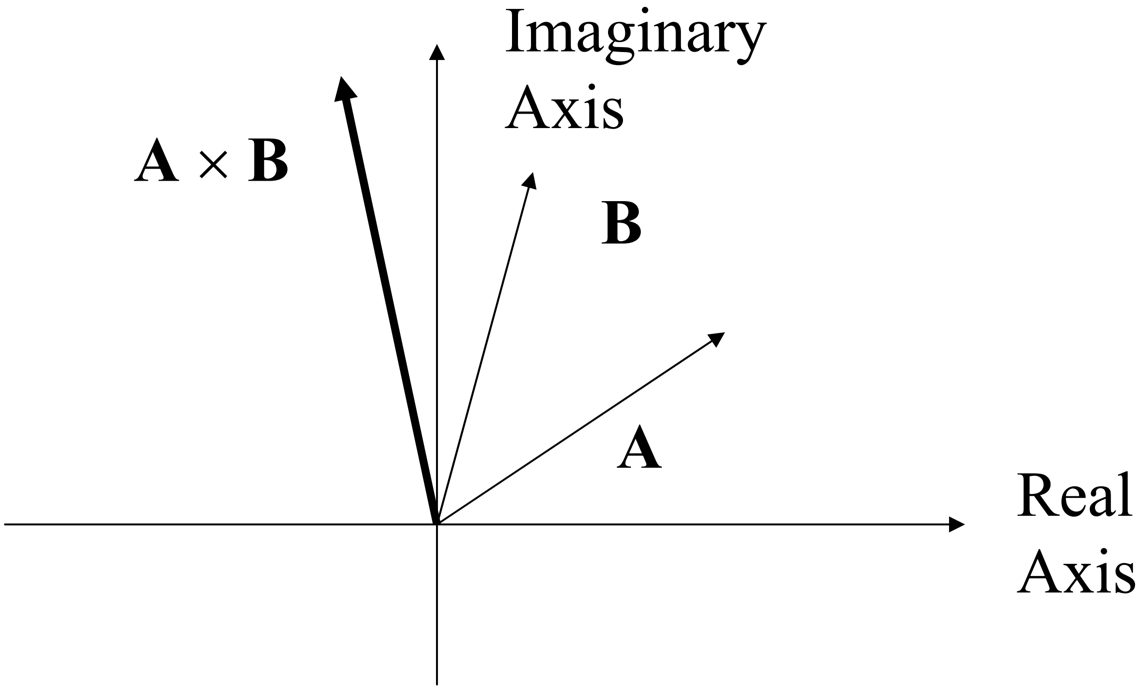
- Multiplication is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} \times \mathbf{B} = (A_M \times B_M) \angle (\theta + \phi)$$

Multiplication



Division

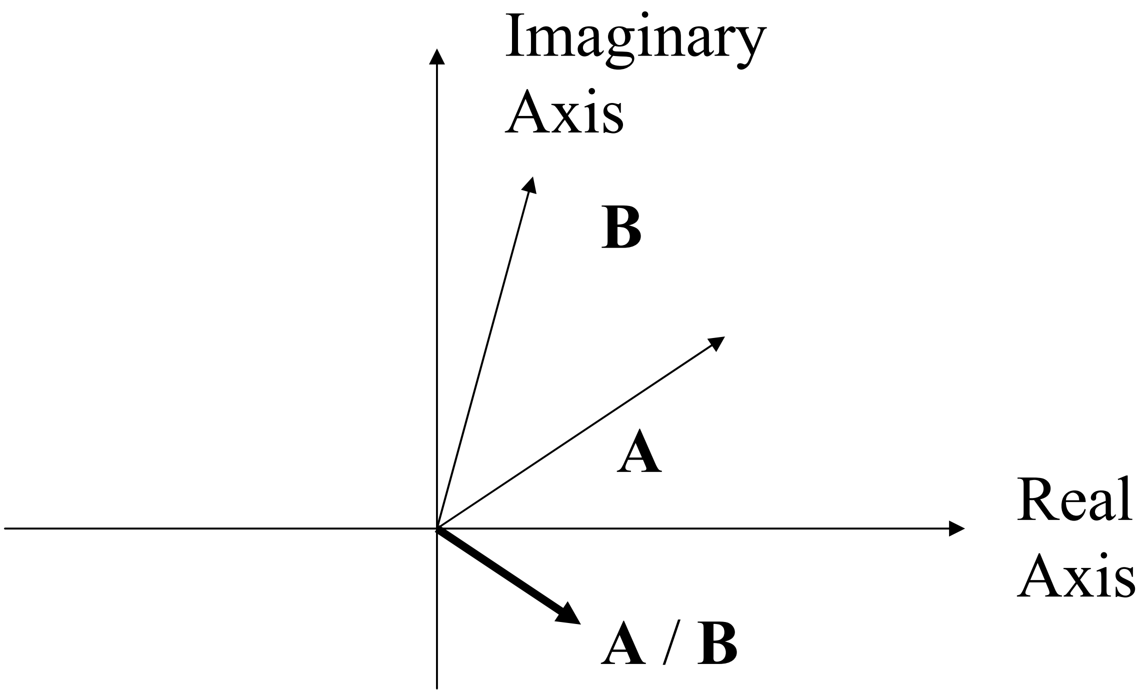
- Division is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} / \mathbf{B} = (A_M / B_M) \angle (\theta - \phi)$$

Division



Phasors

- Assuming a source voltage is a sinusoidal time-varying function

$$v(t) = V\cos(\omega t + \theta)$$

- We can write:

$$v(t) = V\cos(\omega t + \theta) = V\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[Ve^{j(\omega t+\theta)}\right]$$

$$Define \;\; Phasor \;\; as \;\; Ve^{j\theta} = V\angle\theta$$

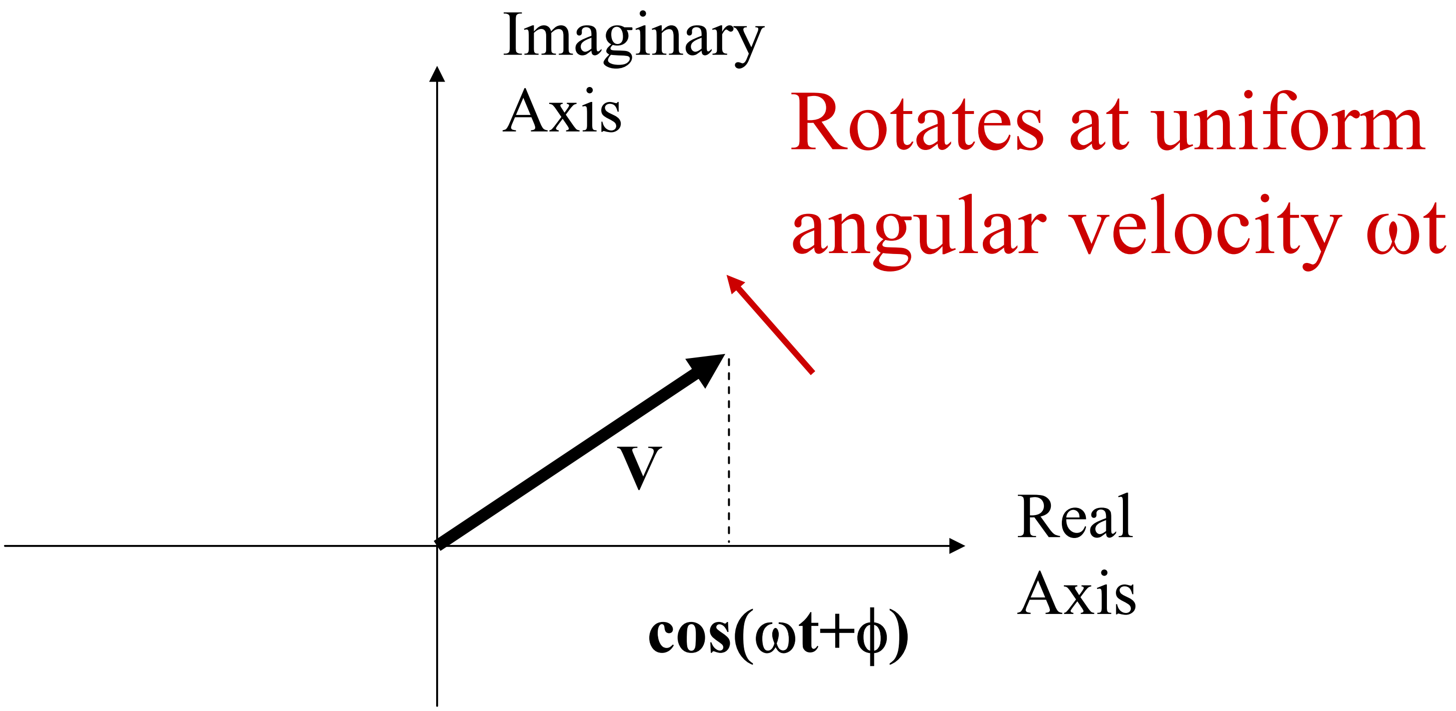
- Similarly, if the function is $v(t) = V\sin(\omega t + \theta)$

$$v(t) = V\sin(\omega t + \theta) = V\cos(\omega t + \theta - \frac{\pi}{2}) = \mathrm{Re}\left[Ve^{j(\omega t+\theta-\frac{\pi}{2})}\right]$$

$$Phasor = V\angle\left(\theta - \frac{\pi}{2}\right)$$

Phasor from rotating Complex Vector

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{Ve^{j\phi}e^{jwt}\right\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right)$$

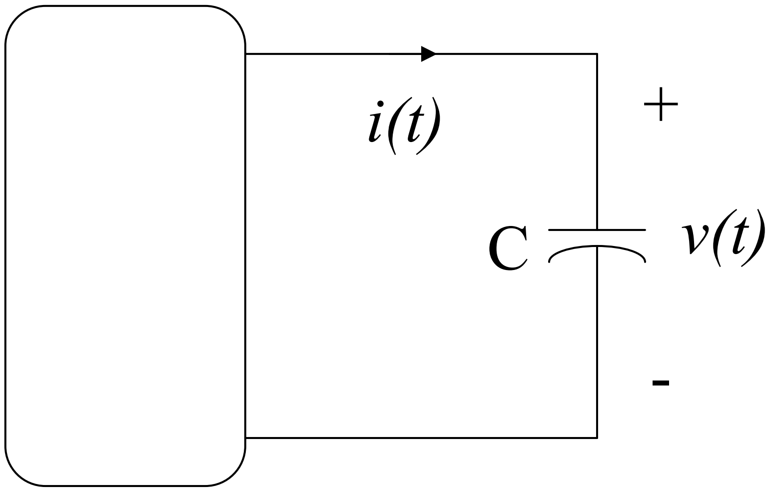


The head start angle is φ.

Complex Exponentials

- We represent a real-valued sinusoid as the real part of a complex exponential after multiplying by $e^{j\omega t}$.
- Complex exponentials
  - provide the link between time functions and phasors.
  - Allow derivatives and integrals to be replaced by multiplying or dividing by $j\omega$
  - make solving for AC steady state simple algebra with complex numbers.
- Phasors allow us to express current-voltage relationships for inductors and capacitors much like we express the current-voltage relationship for a resistor.

I-V **Relationship for a Capacitor**



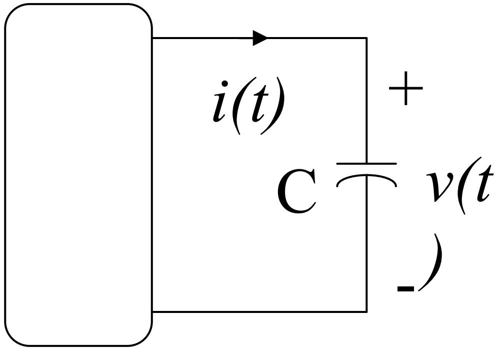
$$i(t) = C\frac{dv(t)}{dt}$$

Suppose that *v(t)* is a sinusoid:

$$v(t) = \mathrm{Re}\{V_M\, e^{j(\omega t+\theta)}\}$$

Find *i(t)*.

Capacitor Impedance (1)



$$i(t) = C\frac{dv(t)}{dt}$$

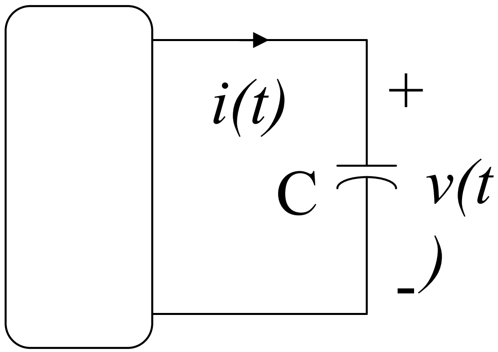
$$v(t) = V\cos(\omega t + \theta) = \frac{V}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right]$$

$$i(t) = C\frac{dv(t)}{dt} = \frac{CV}{2}\frac{d}{dt}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] = \frac{CV}{2}j\omega\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right]$$

$$= \frac{-\omega CV}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right] = -\omega CV\sin(\omega t + \theta) = \omega CV\cos(\omega t + \theta + \frac{\pi}{2})$$

$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\left(\theta + \frac{\pi}{2}\right)} = \frac{V}{\omega CV}\angle(\theta - \theta - \frac{\pi}{2}) = \frac{1}{\omega C}\angle(-\frac{\pi}{2}) = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

Capacitor Impedance (2)



$$i(t) = C\frac{dv(t)}{dt}$$

Phasor definition

$$v(t) = V\cos(\omega t + \theta) = \text{Re}\left[Ve^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{V} = V\angle\theta$$

$$i(t) = C\frac{dv(t)}{dt} = \text{Re}\left[CV\frac{de^{j(\omega t+\theta)}}{dt}\right] = \text{Re}\left[j\omega CVe^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{I} = I\angle\theta$$

$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\theta} = \frac{V}{j\omega CV}\angle(\theta-\theta) = \frac{1}{j\omega C}$$

Example

$$v(t) = 120\text{V} \cos(377t + 30^\circ)$$

$$C = 2\mu\text{F}$$

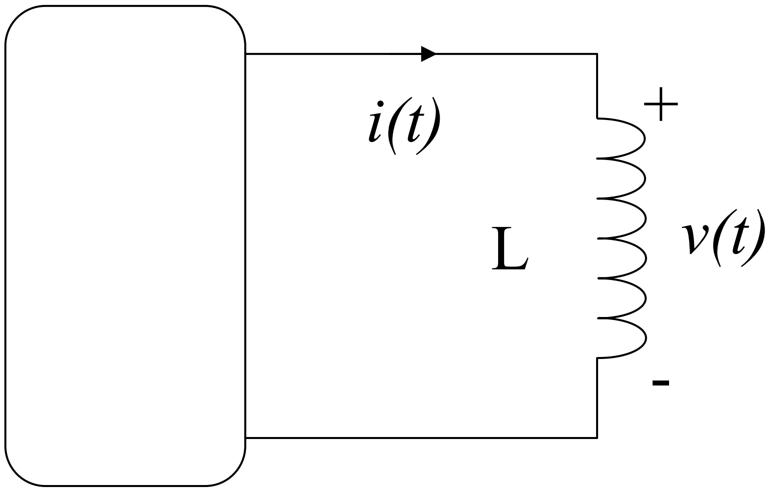
- What is **V**?
- What is **I**?
- What is *i(t)*?

Computing the Current

Note: The differentiation and integration operations become algebraic operations

$$\frac{d}{dt} \Rightarrow j\omega \qquad \int dt \Rightarrow \frac{1}{j\omega}$$

Inductor Impedance



$$v(t) = L\frac{di(t)}{dt}$$

$$\mathbf{V} = j\omega L\,\mathbf{I}$$

Example

$$i(t) = 1\mu A \cos(2\pi\, 9.15\, 10^7 t + 30°)$$

$$L = 1\mu H$$

- What is **I**?
- What is **V**?
- What is *v(t)*?