ELECTRICAL QUANTITIES IN PRESENCE
OF SINUSOIDAL SOURCES WITH THE SAME
FREQUENCY

In a linear system we can use
the superposition theorem. Since a circuit
in a linear system (even in presence of
capacitors and inductors) every electrical
quantity is the weighted sum of
the sources that are present in the circuit.
If the source have all the form:

\[ A \cos(\omega t + \phi) \]

where \( A \) and \( \phi \) can change but \( \omega \) is
always the same (i.e. frequency) then
each quantity has the form

\[ A \cos(\omega t + \phi) \]

where, of course, \( A \) and \( \phi \) are in general
different from the sources. This is true because
we know that a quantity in the circuit
is:

\[ i_x(t) = \sum_i A_i \cos(\omega t + \phi_i) \]
and that same can be reduced to $A\cos(wt+\phi)$ by using Protasenko's formulas. Actually, since we are only interested in $A$ and $\phi$ (magnitude and phase) we could use complex numbers to represent quantities in this kind of situations. We could use the bijection:

$$A\cos(wt+\phi) \rightarrow Ae^{i\phi}$$

This is the phasors method that you will see in introductory sessions.
To understand the transform, let's look at an easy transformation.

Think of a world where we know how to do \( \ln x \) and \( e^x \), also we know how to sum real numbers. Unfortunately multiplication in this world is incredibly expensive: there is a tax for each multiplication you want to do.

We can use a transformed domain. The transformation is the following:

\[
L : \mathbb{R}^+ \rightarrow \mathbb{R} \quad \text{s.t.} \quad x \mapsto \ln x
\]

It is always possible to find \( x \) by using an inverse transformation

\[
y \mapsto e^y
\]

So now if we have to multiply two numbers \( x \cdot y \) we can work in the transformed domain:

\[
\ln(x \cdot y) = \ln x + \ln y = z
\]

Then \( e^z = x \cdot y \) in the inverse transform.
LAPLACE TRANSFORM

It transforms functions of $t$ in functions of a complex variable $s = \alpha + j\omega$

$$\mathcal{L}[f(t)] = \lim_{T\to\infty} \int_0^T f(t) e^{-st} \, dt = F(s)$$

There exists an inverse transformation

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} \, dt$$

where $c$ is an appropriate real value.

This transform has a lot of properties. We note that it is a **linear** transformation

$$\mathcal{L}[Af_1(t) + Bf_2(t)] = AF_1(s) + BF_2(s)$$

We are interested in the derivation and integration theorems:

$$\mathcal{L}\left[ \frac{df(t)}{dt} \right] = sF(s) - f(0)$$
Proof: using integration by parts

\[ \int u'(x)v(x) \, dx = u(x)v(x) - \int u(x)v'(x) \, dx \]

\[ \int \left[ \frac{d}{dt} \left( \int_0^t f(t) \, dt \right) \right] e^{-st} \, dt = \int_0^t f(t) e^{-st} \, dt = \left. f(t)e^{-st} \right|_0^t - \int_0^t f(t)(-se^{-st}) \, dt = \]

\[ f(0) \cdot 0 - f(0) e^0 + sF(s) = sF(s) - f(0) \]

**Corollary:**

\[ \frac{d}{dt} \left( \int_0^t f(t) \, dt \right) = G(s) \]

\[ \frac{d}{dt} \left( \int_0^t f(t) \, dt \right) = F(s) = sG(s) - \int_0^0 f(t) \, dt = \]

\[ sG(s) \Rightarrow G(s) = \frac{F(s)}{s} \]

\[ \frac{d}{dt} \left( \int_0^t f(t) \, dt \right) = \frac{F(s)}{s} \]

1) derivative in time domain is multiplication by s in the s domain.

2) integration in time domain is multiplied by \( \frac{1}{s} \) in the s domain.
TOWARDS DEFINITION OF IMPEDANCE

For a capacitor:
\[ i(t) = C \frac{dV(t)}{dt} \]

Transform each side of the equation:

\[ I(s) = C (sV(s) - V(0)) = CSV(s) - CV(0) \]

If \( V(0) = 0 \) \( \Rightarrow I(s) = CV(s) \) \( \Rightarrow \frac{V(s)}{I(s)} = \frac{1}{sC} \)

For an inductor:
\[ v(t) = L \frac{di(t)}{dt} \quad \Rightarrow \quad V(s) = L (sI(s) - i(0)) = \]
\[ = LI(s) - LI(0) \]

If \( i(0) = 0 \) \( \Rightarrow V(s) = LI(s) \) \( \Rightarrow \frac{V(s)}{I(s)} = sL \)
The quantity \( \frac{V(s)}{I(s)} \) is called **Impedance**.

Depending on the component it may or may not have the same unit of measure of the corresponding quantity in the time domain.

In the case of a resistor:

\[
\frac{V(s)}{I(s)} = R
\]

But for other components, it is not true.
We said that the transformation is linear so we can actually transform KVL and KCL.

Basically these laws hold also in the $s$ domain.

For instance, at a node \( \sum_{i} I_i(s) = 0 \).

\[
\begin{align*}
\text{If you didn't get the importance of the transformation, consider the differential equation:} \\
\frac{d^2 \nu}{dt^2} + a_1 \frac{d\nu}{dt} + a_0 \nu &= 0 \\
\text{If we apply the Laplace transform we get:} \\
a_2 \left[ s^2 \nu(s) - s \nu(0) - \nu'(0) \right] + a_1 (s \nu(s) - \nu(0)) + a_0 \nu(s) &= 0 \\
\nu(s) \left[ a_2 s^2 + a_1 s + a_0 \right] - a_2 s \nu(0) - a_2 \nu'(0) - a_1 \nu(0) &= 0 \\
\nu(s) &= \frac{a_2 s \nu(0) + a_2 \nu'(0) + a_1 \nu(0)}{a_2 s^2 + a_1 s + a_0}
\end{align*}
\]
L transforms differential equations into algebraic equations!!!

Some known transformations:

\[
\sin(wt) \quad \rightarrow \quad \frac{\omega_0}{s^2 + \omega_0^2}
\]

\[
\cos(wt) \quad \rightarrow \quad \frac{s}{s^2 + \omega_0^2}
\]

Step function \[1_{[a,b]} \quad \rightarrow \quad \frac{1}{s}
\]

Dirac's delta \[\delta(t) \quad \rightarrow \quad 1\]

\[e^{-st} \quad \rightarrow \quad \frac{1}{s - s_0}\]

The method

1) Transform all sources (this is easy just use the following equivalences:

\[v(t) \quad \rightarrow \quad \mathcal{L}[v(s)]\]
\[i(t) \quad \rightarrow \quad \mathcal{L}[i(s)]\]
2) Transform capacitor and inductors:

\[ N(t) = \frac{1}{C} \int c(t) \, dt \]

\[ i(t) = C \frac{dN(t)}{dt} \]

\[ I(s) = sC V(s) \]

\[ V(0) \neq 0 \]

\[ I(s) = sC V(s) - V(0) C \]

\[ Y(s) = \frac{I(s)}{sC} + \frac{V(0)}{s} \]

Transform all inductors (you can find the transformation by yourself)

3) Use node-voltage or mesh-current analysis and find all quantities as function of \( s \)

4) Apply \( \mathcal{L}^{-1} \) to find the time domain response of the circuit
Examples:

\[ v(t) \quad i(t) \quad c \]

\[ R \quad \text{N}(t) \text{ in the step function:} \]

\[ N(t) \quad A \quad t \]

Consider also the initial condition \( v(0) = B \).

We use Laplace transform:

\[ V(s) + \frac{1}{sC} \quad i(s) = V(s) + \frac{B}{s} \quad \text{and deal with the capacitor as if it was a resistor whose value is } \frac{1}{sC} \]

\[ -V(s) + RI(s) + B \cdot \frac{1}{s} + I(s) \cdot \frac{B}{sC} = 0 \]
\[ I(s) = \frac{V(s) - B/s}{R + \frac{1}{SC}} = \frac{sV(s) - B}{SR + \frac{1}{C}} \]

Now we can compute \( V_c(s) \) which is not only the voltage across the impedance \( \frac{1}{SC} \) but it also includes the fictitious voltage source that we introduced to take into account the initial condition:

\[ V_c(s) = I(s) \cdot \frac{1}{SC} + \frac{B}{S} = \frac{I(s) + BC}{SC} \]

\[ = \frac{sV(s) - B}{SR + \frac{1}{C}} + BC \]

\[ = \frac{sV(s) + SCR B}{SC} = \frac{sv(s) - B + SCR B}{S^2RC + S} \]

\[ = \frac{sV(s) + SCR B}{S(SC + 1)} \]

\( V(t) \) is a step function with amplitude \( A \) at \( t = 0 \) (by looking at the table on page 9.10)

\[ V(s) = \frac{A}{S} \quad \text{at} \quad t = 0 \]
To find \( v(t) \) we use a method which is call "partial fractions method".

We want to write \( V_c(s) \) in the following form:

\[
V_c(s) = \frac{\alpha}{s} + \frac{\beta}{(SRC+1)}
\]

You can verify that:

\[
\alpha = \lim_{s \to 0} V_c(s) s
\]

\[
\beta = \lim_{s \to -\frac{1}{RC}} V_c(s) (1+SRC)
\]

So we have:

\[
\alpha = A
\]

\[
\beta = \frac{(A + 5CRB)}{s} \bigg|_{s = -\frac{1}{RC}} = (A-B)RC
\]

\[
V_c(s) = \frac{A}{s} + \frac{(B-A)RC}{1 + SRC}
\]

Using the table on page 3.10 we can compute \( L^{-1} \) just by table lookup.
for $t > 0$

$$V(t) = A + (B-A)RC e^{-t/\tau} = A(1-e^{-t/\tau}) + B e^{-t/\tau}$$

where $\tau = RC$.

The result is very interesting. It is the sum of two things: $A(1-e^{-t/\tau})$ is the response to the step function as if the initial condition $V(0)$ was equal to zero. The other term $B e^{-t/\tau}$ is the response to the initial condition.
ANALYSIS IN THE FREQUENCY DOMAIN

We have already seen that if all sources are of the form $A \cos(\omega t + \phi)$ then also any other quantity is of the same form (pog 3.1).

Consider now the voltage source:

$$e(t) = A \cos(\omega t + \phi)$$

Using complex numbers, it can be re-written as:

$$e(t) = A \frac{e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}}{2} = \frac{A}{2} \left[ e^{j\omega t} + e^{-j\omega t} \right] - j\phi \left[ e^{j\omega t} - e^{-j\omega t} \right] = \frac{1}{2} \left[ Ae^{j\omega t} + Ae^{-j\omega t} \right] = \frac{1}{2} \left[ E e^{j\omega t} + E^* e^{-j\omega t} \right]$$

where $E = A e^{j\phi}$ and $E^*$ is the complex conjugate of $E$. $E$ is called phasor of $e(t)$. Now we can take the Laplace transform of $e(t)$.
\[ \mathcal{L}[e(t)] = \mathcal{L} \left[ \frac{E}{2} e^{j\omega t} + \frac{E^*}{2} e^{-j\omega t} \right] = \]
\[= \frac{1}{2} \left[ \frac{E}{s-j\omega_0} + \frac{E^*}{s+j\omega_0} \right] = E(s) \]

Consider now another quantity that we are interested in (maybe a voltage across a capacitor for instance), say \( u(t) \). Its Laplace transform is \( U(s) \).

\( E(s) \) and \( U(s) \) are related by a function of \( s \) that can be computed by solving the circuit. So we can write:

\[ U(s) = F(s) E(s) \quad \text{(} F(s) \text{ is called transfer function)} \]

But we also know that \( u(t) = A \cos(\omega t + \phi) \) so its transform is:

\[ U(s) = \frac{1}{2} \left[ \frac{U}{s-j\omega_0} + \frac{U^*}{s+j\omega_0} \right] \]

Where \( U \) is the phase of \( u(t) \), which is basically \( A e^{j\phi} \).
If we want to compute $U$, we can use the same method that we used for partial fractions (p. 9.14):

\[
U = \lim_{s \to j\omega} 2 \frac{U(s)}{s-j\omega} = \lim_{s \to j\omega} 2 \frac{F(s)}{2} \left[ \frac{E}{s-j\omega} + \frac{E^*}{s+j\omega} \right] (s-j\omega) = \lim_{s \to j\omega} F(s) \left[ E + \frac{E^* (s-j\omega)}{s+j\omega} \right] = F(j\omega) E
\]

$F(j\omega)$ is a complex number and can hence be represented as: $F(j\omega) = |F(j\omega)| e^{j\phi F(j\omega)}$

So the phase $U = |F(j\omega)| e^{j\phi F(j\omega)} \cos(\omega t + \theta + \phi F(j\omega))$

\[
\text{and } \quad u(t) = A \left[ \frac{1}{A'} \right] \cos(\omega t + \theta + \phi F(j\omega)) \]

\[
\text{A'}
\]

\[
\theta
\]
It is important to notice that $F(j\omega)$ is the Fourier transform of the transfer function $\frac{u(t)}{e(t)}$ in $\omega$.

If we leave $\omega$ to be a variable, then we can understand how our circuit behaves at different frequencies.

Example:

![Electrical Circuit Diagram]

We want to compute the frequency response of the circuit, or better the steady state sinusoidal behavior. To do this, we consider all initial conditions equal to zero (steady state).
and we compute the transfer function

$$F(s) = \frac{V_c(s)}{V(s)}$$

$$F(j\omega)$$ says \( \forall \omega \), how the circuits modifies the amplitude and phase of a sinusoidal input \( v(t) \).

$$V_c(s) = V(s) \frac{V_{sc}}{R + \frac{1}{V_{sc}}} = V(s) \frac{1}{1 + SRC}$$

$$\frac{V_c(s)}{V(s)} = \frac{1}{1 + SRC} = F(s)$$

$$F(j\omega) = \frac{1}{1 + j\omega RC}$$

$$|F(j\omega)| = \left| \frac{1 - j\omega RC}{(1 + j\omega RC)(1 - j\omega RC)} \right| = \frac{|1 - j\omega RC|}{1 + (\omega RC)^2} = \frac{\sqrt{1 + (\omega RC)^2}}{1 + (\omega RC)^2} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$
\[ \delta F(jw) = \tan^{-1}\left( \frac{\text{Im}[F(jw)]}{\text{Re}[F(jw)]} \right) = \tan^{-1}(-\omega RC) \]

This means that if \( v(t) = A\cos(\omega t) \)
then
\[ V_c(t) = \frac{A}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t + \tan^{-1}(-\omega RC)) \]

The phase sometimes could be important, but the change in the amplitude is very important because depending on the frequency the output could become very small.