NODE/LOOP ANALYSIS

Lecture 3 review:

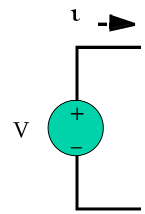
- Voltage and current sources
- Series and parallel resistors

Today:

- Nodal analysis
- Loop analysis
- Series & parallel resistors (revisit & generalize)
- Real voltmeters
- Real ammeters

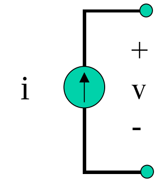
CURRENT-VOLTAGE CHARACTERISTICS OF VOLTAGE & CURRENT SOURCES

Describe a two-terminal circuit element by plotting current vs. voltage



Ideal voltage source

We assume *un*associated signs so i comes out of + terminal



Ideal current source

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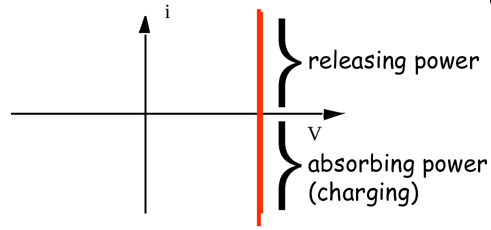
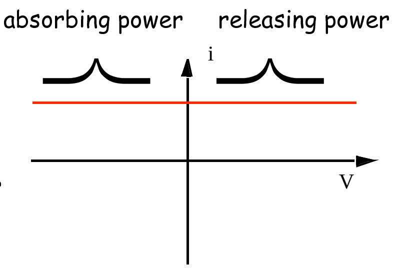
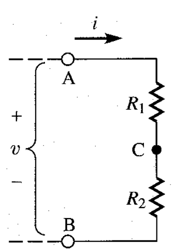
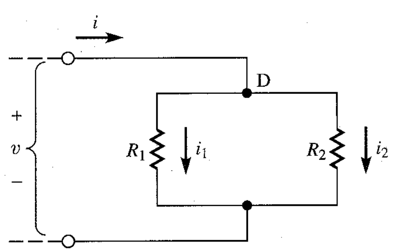




Resistors in Parallel and Series



Series



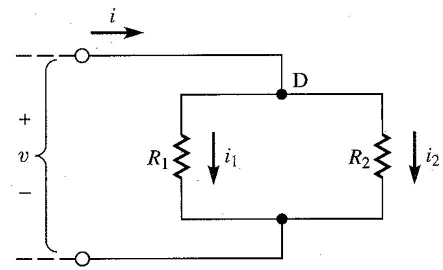
Parallel

In series:

$v = i R_1 + i R_2 = i(R_1+R_2) = i(R)$, where $R=R_1+R_2$

continue

- **In Parallel**



Apply KCL at node D

$i = i_1 + i_2 = (v/R_1 - v/R_2)$

$= v\,(1/R_1 - 1/R_2)$

$= v\,(R_2 + R_1)/(R_1R_2)$

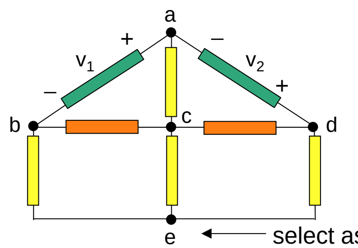
$= v\,/[(R_1R_2)/(R_2 + R_1)]$

$= v\,/R$

$R = [(R_1R_2)/(R_2 + R_1)]$, where R_1 is in parallel with R_2

BRANCH AND NODE VOLTAGES

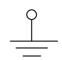
The voltage across a circuit element is defined as the difference between the node voltages at its terminals



$v_2 = v_d - v_a$

$v_e \equiv 0$ (since it's the reference)

← select as ref. ⇒



"ground"

Specifying node voltages: Use one node as the implicit reference (the "common" node … attach special symbol to label it)

Now single subscripts can label voltages:

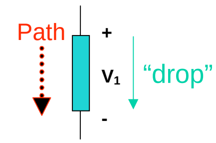
e.g., v_b means $v_b - v_e$, v_a means $v_a - v_e$, etc.

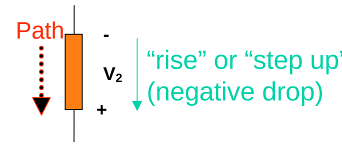
KIRCHHOFF'S VOLTAGE LAW (KVL)

The algebraic sum of the "voltage drops" around any "closed loop" is zero.

Why? We must return to the same potential (conservation of energy).

Voltage drop → defined as the branch voltage if the + sign is encountered first; it is (-) the branch voltage if the – sign is encountered first … important bookkeeping



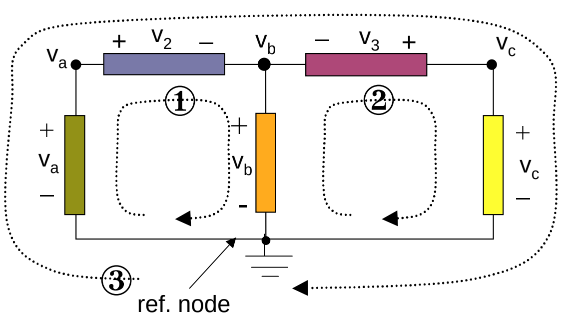


Closed loop: Path beginning and ending on the same node

KVL EXAMPLE

Examples of
Three closed paths:

①, ②, ③



Note that:

$v_2 = v_a - v_b$

$v_3 = v_c - v_b$

Path 1:

$-v_a + v_2 + v_b = 0$

↑

$v_a - v_b$

YEP!

Path 2:

$-v_b - v_3 + v_c = 0$

Path 3:

$-v_a + v_2 - v_3 + v_c = 0$

ALTERNATIVE STATEMENTS OF KIRCHHOFF'S VOLTAGE LAW

1. For any node sequence A, B, C, D, …, M around a closed path, the voltage drop from A to M is given by

$$v_{AM} = v_{AB} + v_{BC} + v_{CD} + \cdots + v_{LM}$$

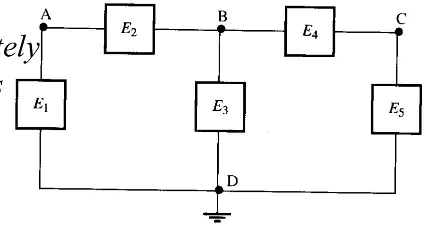
2. For all pairs of nodes i and j, the voltage drop from i to j is

$$v_{ij} = v_i - v_j$$

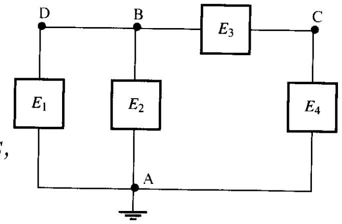
where the node voltages are measured with respect to the common node.

Node Analysis

- *The state of a circuit is completely known if the voltage at all nodes & and the current thru all the branches is known.*
- *One of the node voltages can be arbitrarily set to zero.*
- *The total unknowns to be solved for is equal to the number of branches plus the number of nodes, minus one.*



(a)



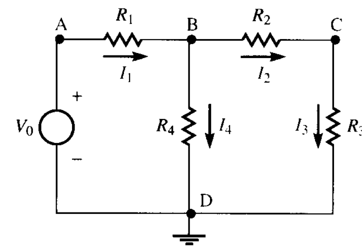
(b)

Node Analysis

Goals: *Solve all unknown node voltages.*

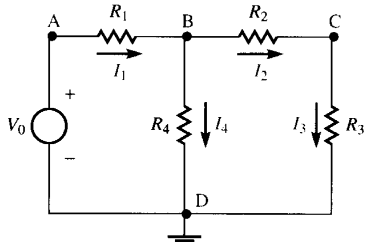
How?

- *By writing equations expressing Kirchhoff's current law (KCL) for each node where the voltage is unknown.*



Node Analysis (cont.)

5 circuit elements - 4 resistors, 1 voltage source.
4 nodes:
- *2 unknown voltages:* v_B, v_C
- *2 known voltages:*
  - *1 reference voltage:* v_D
  - *1 voltage source:* $v_A = v_o$



4 branch currents: I_1, I_2, I_3, I_4
(but $I_2 = I_3$*) => 3 unknown currents*

Apply KCL at node B:

$$\frac{V_0 - V_B}{R_1} + \frac{V_C - V_B}{R_2} + \frac{V_D - V_B}{R_4} = 0$$

Apply KCL at node C:

$$\frac{V_B - V_C}{R_2} + \frac{V_D - V_C}{R_3} = 0$$

But $V_D = 0$ in both cases.

Node Analysis

$$\frac{V_0 - V_B}{R_1} + \frac{V_C - V_B}{R_2} + \frac{-V_B}{R_4} = 0 \qquad \frac{V_B - V_C}{R_1} + \frac{-V_C}{R_3} = 0$$

2 equations and 2 unknowns.
Can be solved simultaneously for V_B *&* V_C

$$V_C = V_0 \cdot \frac{R_3 R_4}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4}$$

$$V_B = V_0 \cdot \frac{R_4 (R_2 + R_3)}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4}$$

$$\left.\begin{array}{l} V_B = f(V_0) \\ V_C = f(V_0) \end{array}\right\}$$

$$if \begin{cases} V_0 = 0 \\ V_B = V_C = 0 \end{cases}$$

What about the 4 branch currents?
Apply Ohm's law

$$I_1 = \frac{V_A - V_B}{R_1} = \frac{V_0 - V_B}{R_1} \quad I_2 = \frac{V_B - V_C}{R_2} \quad I_3 = \frac{V_C - V_D}{R_3} = \frac{V_C}{R_3} \quad I_4 = \frac{V_B - V_D}{R_4} = \frac{V_B}{R_4}$$

Node Analysis (nodal analysis)

1 Choose a Reference Node ⏚

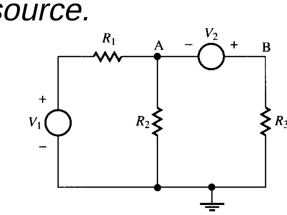
2 Define unknown node voltages (those not fixed by voltage sources)

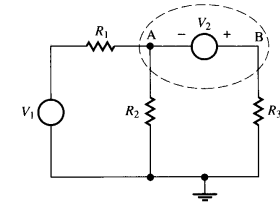
3 Write KCL at each unknown node, expressing current in terms of the node voltages (using the constitutive relationships of branch elements: **for resistor element use Ohm's law**)

4 Solve the set of equations (N equations for N unknown node voltages)

Node Analysis (special case)

When 2 nodes (voltages are unknown) are connected by a voltage source.



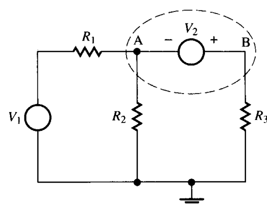


For the circuit on the left, we can not apply KCL at node A using the normal approach. We can not write the current flowing from A to B, thus, can not find the sum of current entering node A. Similar problem if we apply KCL at node B.

Circumvent this by drawing a dashed line around the 2 nodes as shown in the circuit on the right. Consider it as a "supernode", and apply to same KCL on this "supernode".

Node Analysis (special case (cont.))

=> sum of all current entering this "supernode" or the dashed line is zero:



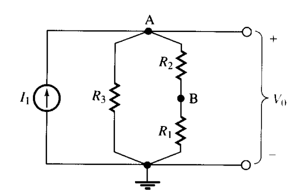
$$\frac{V_1 - V_A}{R_1} + \frac{0 - V_A}{R_2} + \frac{0 - V_B}{R_3} = 0$$

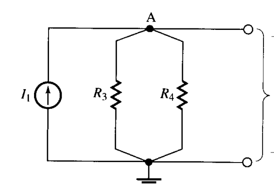
$$V_A + V_2 = V_B$$

Remember V_A gain V_2 (from - to +) yields V_B.

- *Remember N equations and N unknowns.*
- *In this case, 2 equations and 2 unknowns would allow us to solve for the 2 unknowns simultaneously.*

Node Analysis (example)





Find V_0 in the circuit on the left.

First, simplify the circuit by replacing the series combination of R_1 and R_2 by R_4 ($R_4 = R_1 + R_2$)

Now there is only one unknown node voltage left : $V_A = V_0$

Apply KCL at node A (summing all the current entering node A):

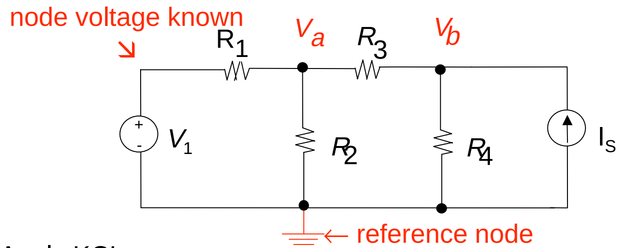
$$I_1 + \frac{0 - V_A}{R_3} + \frac{0 - V_A}{R_4} = 0$$

One equation and one unknown

$$V_A = I_1 \frac{R_3 R_4}{R_3 + R_4} = I_1 \frac{R_3 (R_1 + R_2)}{R_3 + (R_1 + R_2)}$$

EXAMPLE OF NODE ANALYSIS

Define the node voltages (except reference node and the one set by the voltage source); write down set of equations for node voltages V_a and V_b



Apply KCL:

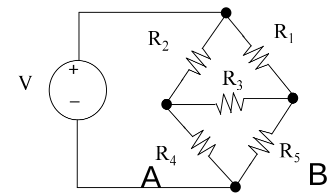
$$\frac{V_a - V_1}{R_1} + \frac{V_a - V_b}{R_3} + \frac{V_a}{R_2} = 0$$

$$\frac{V_b - V_a}{R_3} + \frac{V_b}{R_4} - I_S = 0$$

You can solve for V_a, V_b.

What if we used different ref node?

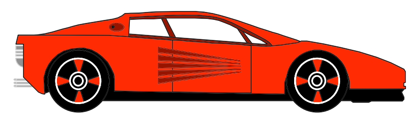
Closed/Open circuit



Short circuit A and B means $R_3 = 0$ (current can go thru, usually a lot of current)

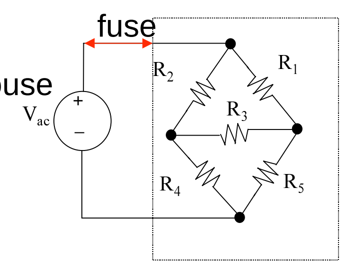
Open circuit A and B means $R_3 = \infty$ (no current can go thru)

Why is electric window not a "safe" option in a car?



What happen if some device is short circuit/open circuit in our house?

Power from outside the house



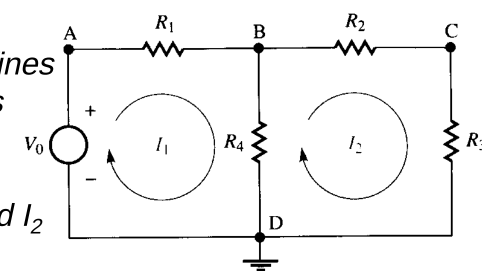
House

Short circuit within the house => draw a lot of current=> fuse would be melted at higher than a certain temperature (or current) => The electric system in your house becomes open circuit (disconnected). (alternate, a breaker can be switch off during a short circuit). Could also happen if the electric load in the house is too high, ie. extension cord is used by several electric appliance at the same time.

Loop Analysis

- *In node analysis (or nodal analysis), voltage is obtained first by using KCL, then current is found by using the V-I characteristic of the circuit element (for resistor element: Ohm's law).*
- *In loop analysis, current is obtained first by KVL, then voltage is found by using the V-I characteristic (for resistor element: Ohm's law).*

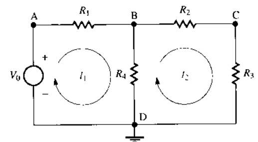
In loop analysis, one defines special current known as ***mesh currents.***
In this example, 2 mesh currents is defined, I_1 and I_2



Loop Analysis (Mesh Analysis)

The number of mesh currents = (Number of branches) - (Number of nodes) + 1

In this case, there are 4 nodes, 5 branches.
# of mesh = 5 - 4 + 1 = 2



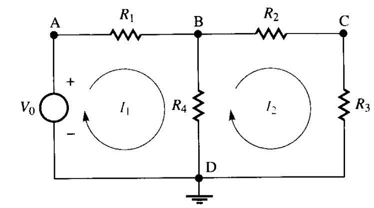
How to define mesh current?
A lot of freedom, except that every branch of the circuit have at least one mesh current flowing thru it.

The value of the mesh currents are now the unknown to be solved for. The number of equations = The number of unknown mesh current

***The equations** are obtained by applying Kirchhoff Voltage law on the loops of the mesh currents.*

Loop Analysis

Assume I_1 and I_2 are defined as the mesh current. Notice the mesh current go thru the whole loop. R_4 has 2 mesh currents go thru it.



The voltage drop from A to B across R_1: $I_1 R_1$

The voltage drop from B to D across R4: $(I_1 - I_2)R_4$

Apply KVL in the loop ABDA:

$$I_1 R_1 + (I_1 - I_2)R_4 - V_0 = 0$$

Apply KVL in the loop BCDB:

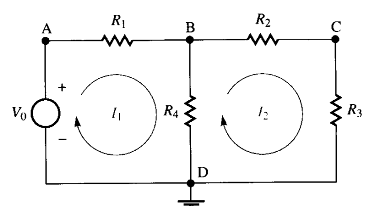
$$I_2 R_2 + I_2 R_3 + (I_2 - I_1)R_4 = 0$$

Loop Analysis

Solving these 2 equations simultaneously:

$$I_1 = V_0 \frac{R_2 + R_3 + R_4}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4}$$

$$I_2 = V_0 \frac{R_4}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4}$$



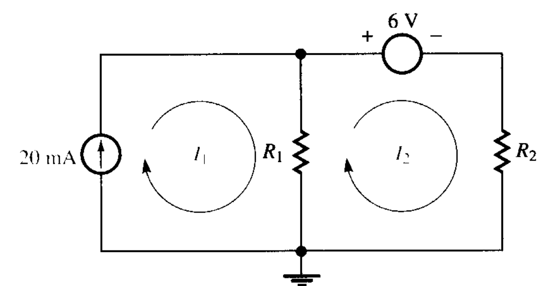
The individual branch current and voltage can be found from I_1 and I_2.

$$V_{CD} = V_C = I_2 R_3 = V_0 \frac{R_4 R_3}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4}$$

Loop Analysis

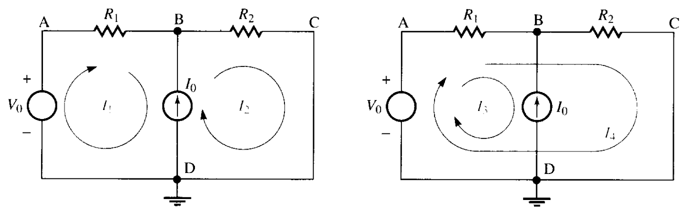
Need as many loop equations as there are unknown mesh current.

In this case, I_1 is known because it is equal to 20mA. So there is only one mesh current as unknown in this circuit.



Loop Analysis (special case)

Special case when 2 unknown mesh currents both pass thru a current source like the circuit on the left.



It is impossible to write KVL for the path ABDA because one cannot write the voltage drop across a current source as a function of the current thru it.
To circumvent this, we use I_3 and I_4 as mesh currents as shown on the right circuit.

Notice: $I_3 = -I_0$, so the only unknown mesh current is I_4.
And I_4 can be found by using the normal KVL approach.

Loop Analysis (mesh analysis)

1. *Select the **proper number of mesh current** such that at least one mesh current passes through each branch.*
2. *(a) **Express voltage drop** across each element **as functions of known and unknown mesh currents**,*
   *(b) write equations stating that the sum of the voltage drop around closed path are zero. (**KVL**)*
3. ***Solve equations** obtained in step 2 simultaneously for unknown mesh current.*
4. *Obtain branch currents from the mesh current found in in step 3 and obtain desired node voltages from branch currents and the I-V relationship of the branches.*

ANOTHER NOTE ON SUBSCRIPTS

Although the text does not follow this convention, a simple convention exists to identify fixed voltage and current sources. A fixed voltage or current source typically is denoted by double subscripts, e.g., V_{CC}, V_{DD}, etc., and is all capitals. Thus:

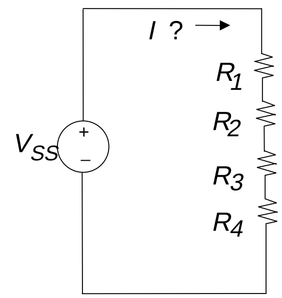
V_{SS}	Fixed voltage (e.g., 5V source)
V_s	DC voltage (may be an unknown)
v_s	Time-varying voltage
I_{BB}	Fixed current (e.g., 1µA source)
i_b	Time-varying current
I_b	dc current

Thus we are certain that all-cap double-subscripted symbols are fixed values. Single-subscripted symbols will be variables when double subscripted symbols are present. Otherwise we have to figure out the symbol type from instructions or context.

RESISTORS IN SERIES

(Here its more convenient to use KVL than node analysis)

Circuit with several resistors in series – Can we find an equivalent resistance?



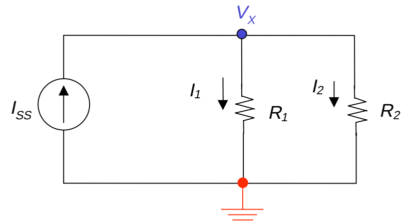
- KCL tells us same current flows through every resistor
- KVL tells us
  $I \cdot R_1 + I \cdot R_2 + I \cdot R_3 + I \cdot R_4 = V_{SS}$
- Clearly,
  $I = V_{SS}/(R_1 + R_2 + R_3 + R_4)$

☞ Thus, equivalent resistance of resistors in series is the simple sum

RESISTORS IN PARALLEL

1 Select Reference Node

2 Define unknown node voltages



Note: $I_{ss} = I_1 + I_2$, i.e.,

$$I_{SS} = \frac{V_X}{R_1} + \frac{V_X}{R_2} \quad \Rightarrow V_X = I_{SS} \cdot \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \quad = I_{SS} \cdot \frac{R_1 R_2}{R_1 + R_2}$$

RESULT 1 EQUIVALENT RESISTANCE: $R_{\|} \equiv R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$

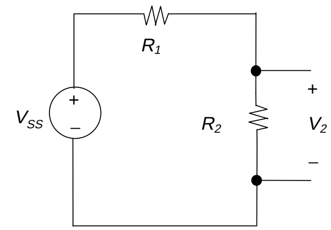
RESULT 2 CURRENT DIVIDER: $I_1 = \frac{V_X}{R_1} = I_{SS} \times \frac{R_2}{R_1 + R_2}$

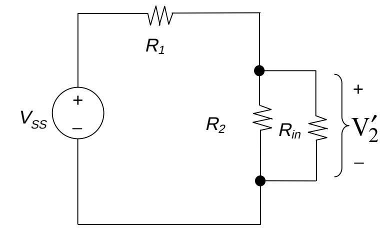
$$I_2 = \frac{V_X}{R_2} = I_{SS} \times \frac{R_1}{R_1 + R_2}$$

REAL VOLTMETERS
Concept of "Loading" as Application of Parallel Resistors

How is voltage measured? Modern answer: Digital multimeter (DMM)

Problem: Connecting leads from voltmeter across two nodes changes the circuit. The voltmeter is characterized by how much current it draws at a given voltage → "voltmeter input resistance," R_{in}. Typical value: 10 MΩ





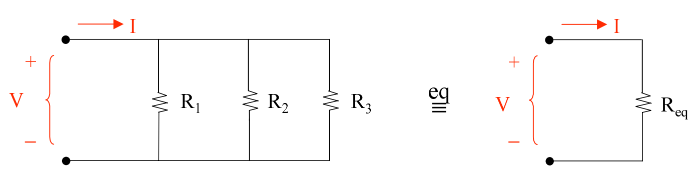
$$V_2 = V_{SS}\left[\frac{R_2}{R_1 + R_2}\right] \qquad V_2' = V_{SS}\left[\frac{R_2 \| R_{in}}{R_2 \| R_{in} + R_1}\right]$$

Example: $V_{SS} = 10\text{V}, R_2 = 100\text{K}, R_1 = 900\text{K} \Rightarrow V_2 = 1\text{V}$

But if $R_{in} = 10\text{M}, V_2' = 0.991\text{V}$, a 1% error

GENERALIZED PARALLEL RESISTORS

What single resistance R_{eq} is equivalent to three resistors in parallel?



$$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \equiv \frac{V}{R_{eq}} \quad \Rightarrow R_{eq} \equiv R_1 \| R_2 \| R_3 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

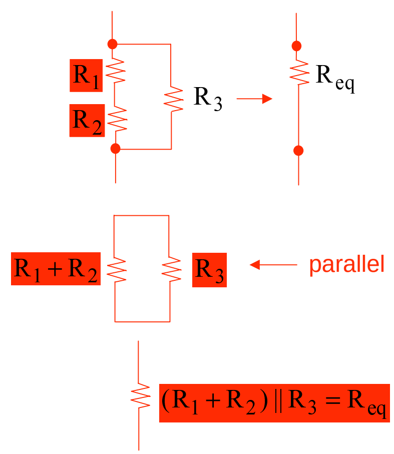
Note the simplicity if we use conductance instead of resistance

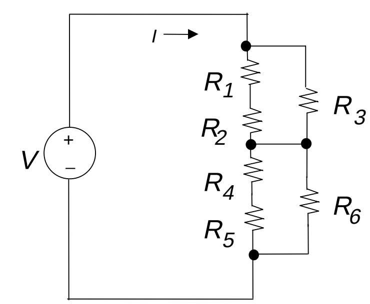
$$G_1 \equiv \frac{1}{R_1}, \text{etc.}, G_{eq} \equiv \frac{1}{R_{eq}}$$

Then, $G_{eq} = G_1 + G_2 + G_3$ ADD CONDUCTANCES IN PARALLEL

IDENTIFYING SERIES AND PARALLEL COMBINATIONS

Use series/parallel equivalents to simplify a circuit before starting KVL/KCL



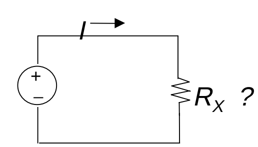


$R_1 = R_2 = 10\text{ K}$

$R_3 = 20\text{ K}$

$R_4 = 5 = 5\text{ K}$

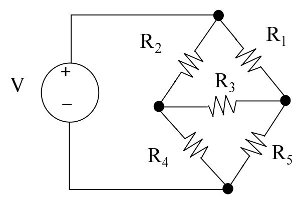
$R_6 = 10\text{ K}$

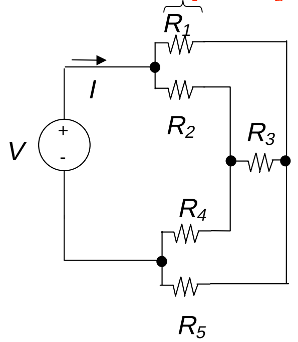


$$R_X = (R_1 + R_2) \| R_3 + (R_4 + R_5) \| R_6 = 15\text{ K}$$

IDENTIFYING SERIES AND PARALLEL COMBINATIONS (cont.)

Some circuits *must* be analyzed (not amenable to simple inspection)





R_1 and R_2 are **not** in ||

R_1 and R_5 are **not** in series

Special cases:
$R_3 = 0$ **OR** $R_3 = \infty$

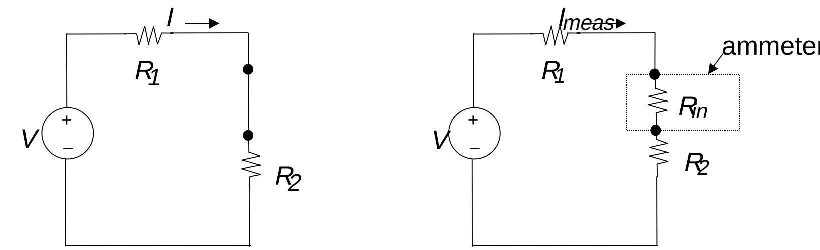
Example: $R_3 = 0 \Rightarrow R_1 \,||\, R_2$; $R_4 \,||\, R_5$ in series; $R_{eq} = R_1 \,||\, R_2 + R_4 \,||\, R_5$

OR IF $R_3 = \infty \Rightarrow (R_1 + R_5) \,||\, (R_2 + R_4)$

MEASURING CURRENT

Insert DMM (in current measurement mode) into circuit. But ammeters disturb the circuit. (Note: Ammeters are characterized by their "ammeter input resistance," R_{in}. Ideally this should be very low. Typical value (in mA range) 1Ω.)

Potential measurement error due to non-zero input resistance:



undisturbed circuit

$$I = \frac{V_1}{R_1 + R_2}$$

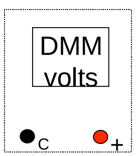
with ammeter

$$I_{meas} = \frac{V_1}{R_1 + R_2 + R_{in}}$$

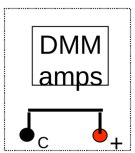
Example V = 1 V: R1 + R2 = 1 KΩ , Rin = 1Ω

$$I = 1\,\text{mA} \quad , \quad I_{meas} = \frac{1}{1K + 1\Omega} \cong 0.999\,\text{mA} \quad (0.1\%\ \text{error})$$

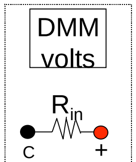
IDEAL AND NON-IDEAL METERS



IDEAL



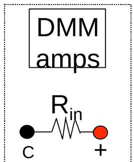
IDEAL



MODEL OF REAL DIGITAL VOLTMETER

Note: R_{in} may depend on range

R_{in} typically $\gtrsim$ 10 MΩ



MODEL OF REAL DIGITAL AMMETER

Note: R_{in} usually depends on current range

R_{in} typically $\lesssim$ 1 Ω