Thévenin & Norton Equivalent Circuits

Lecture 5 review:
• Voltage divider & current divider
• Equivalent circuits (Thévenin & Norton equivalents)

Today: (3.3, 3.4)
• Power Calculations
• Starting your car (model)

RESISTORS IN SERIES
(Here its more convenient to use KVL than node analysis)

Circuit with several resistors in series – Can we find an equivalent resistance?

KCL tells us same current flows through every resistor
KVL tells us
\[ I \cdot R_1 + I \cdot R_2 + I \cdot R_3 + I \cdot R_4 = V_{SS} \]

Thus, equivalent resistance of resistors in series is the simple sum

GENERALIZED VOLTAGE DIVIDER

Circuit with several resistors in series

We know
\[ I = V_{SS} \frac{R_2}{R_1 + R_2 + R_3 + R_4} \]

Thus,
\[ V_1 = \frac{R_1}{R_1 + R_2 + R_3 + R_4} \cdot V_{SS} \]

and
\[ V_3 = \frac{R_3}{R_1 + R_2 + R_3 + R_4} \cdot V_{SS} \]

etc.. etc..

WHEN IS VOLTAGE DIVIDER FORMULA CORRECT?

Correct if nothing else connected to nodes

What is \( V_2 \)?
Answer:
\[ V_2 = \frac{R_2}{R_1 + R_2 + R_5(R_3 + R_4)} \cdot V_{SS} \]
RESISTORS IN PARALLEL

1 Select Reference Node
2 Define unknown node voltages

Note: \( I_{SS} = I_1 + I_2 \), i.e.,

\[
I_{SS} = \frac{V_X}{R_1} + \frac{V_X}{R_2} \Rightarrow V_X = I_{SS} \cdot \frac{R_1 R_2}{R_1 + R_2}
\]

RESULT 1 EQUIVALENT RESISTANCE: \( R_{||} = R_1 || R_2 = \frac{R_1 R_2}{R_1 + R_2} \)

RESULT 2 CURRENT DIVIDER:

\[
I_1 = \frac{V_X}{R_1} = I_{SS} \times \frac{R_2}{R_1 + R_2}
I_2 = \frac{V_X}{R_2} = I_{SS} \times \frac{R_1}{R_1 + R_2}
\]

REAL VOLTMETERS

Concept of “Loading” as Application of Parallel Resistors

How is voltage measured? Modern answer: Digital multimeter (DMM)

Problem: Connecting leads from voltmeter across two nodes changes the circuit. The voltmeter is characterized by how much current it draws at a given voltage → “voltmeter input resistance,” \( R_{in} \). Typical value: 10 MΩ

\[
V_2 = V_{SS} \left[ \frac{R_2}{R_1 + R_2} \right]
V_2 = V_{SS} \left[ \frac{R_2}{R_2 || R_{in}} \right]
\]

Example: \( V_{SS} = 10V, R_2 = 100K, R_1 = 900K \) ⇒ \( V_2 = 1V \)

But if \( R_{in} = 10M \), \( V_2' = 0.991V \), a 1% error

GENERALIZED PARALLEL RESISTORS

What single resistance \( R_{eq} \) is equivalent to three resistors in parallel?

\[
I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \equiv \frac{V}{R_{eq}} \Rightarrow R_{eq} = R_1 || R_2 || R_3 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}
\]

Note the simplicity if we use conductance instead of resistance

\[
G_1 = \frac{1}{R_1} \text{ etc., } G_{eq} = \frac{1}{R_{eq}}
\]

Then, \( G_{eq} = G_1 + G_2 + G_3 \)

ADD CONDUCTANCES IN PARALLEL

GENERALIZED CURRENT DIVIDER

Current splits among M resistors in parallel

Formal approach:

\[
V = \left( \frac{1}{R_1} \right) + \left( \frac{1}{R_2} \right) + \left( \frac{1}{R_3} \right)
I_3 = \left[ \frac{1}{R_1 + 1/R_2 + 1/R_3} \right] \Rightarrow I_3 = \left[ \frac{G_3}{G_1 + G_2 + G_3} \right]
\]

Note \( I_3 = \left[ \frac{G_3}{G_1 + G_2 + G_3} \right] \)

Can we get this result by inspection?

YES – Consider \( R_1 || R_2 \) as “one equivalent resistor”

Then

\[
I_3 = \left( \frac{R_1 || R_2}{R_1 || R_2 + R_3} \right) \times I
\]
IDENTIFYING SERIES AND PARALLEL COMBINATIONS
Use series/parallel equivalents to simplify a circuit before starting KVL/KCL

R_1 = R_2 = 10 K
R_3 R_3 = 20 K
R_4 = 5 = 5 K
R_6 = 10 K

\[(R_1 + R_2) || R_3 = R_{eq}\]

\[R_X = (R_1 + R_2) || R_3 + (R_4 + R_5) || R_6 = 15 K\]

Some circuits must be analyzed (not amenable to simple inspection)

R_1 and R_2 are not in ||
R_4 and R_5 are not in series

Special cases:
R_3 = 0 OR R_3 = \infty

Example: R_3 = 0 \Rightarrow R_1 || R_2; R_4 || R_5 in series;
OR IF R_3 = \infty \Rightarrow (R_1 + R_2) || (R_2 + R_4)

MEASURING CURRENT
Insert DMM (in current measurement mode) into circuit. But ammeters disturb the circuit. (Note: Ammeters are characterized by their "ammeter input resistance," R_{in}. Ideally this should be very low. Typical value (in mA range) 1 Ω.)

Potential measurement error due to non-zero input resistance:

\[I = \frac{V_i}{R_1 + R_2}\]
\[I_{meas} = \frac{V_i}{R_1 + R_2 + R_{in}}\]

Example V = 1 V: R1 + R2 = 1 KΩ, R_{in} = 1 Ω
I = 1 mA, \[I_{meas} = \frac{1}{1 K + 1} \approx 0.999 \text{ mA} \quad (0.1\% \text{ error})\]

IDEL AND NON-IDEAL METERS

DMM volts
IDEAL

DMM amps
IDEAL

MODEL OF REAL DIGITAL VOLTMETER
Note: R_{in} may depend on range
R_{in} typically > 10 MΩ

MODEL OF REAL DIGITAL AMMETER
Note: R_{in} usually depends on current range
R_{in} typically < 1 Ω
Equivalent Circuits

Circuits on the left side is equivalent to the circuit on the right. Both circuits have a V-I curve equivalent to a 11kΩ resistor.

Composite Circuit Element

Find the I-V relationship of the circuit on the right.

KCL equation at node X:
\[ 10^{-2} + \frac{V_Y - V_X}{100} + I = 0 \]
\[ V_X = V_Y + 100I + 1 \]

V is equal to:
\[ V = V_{X,Y} + 1 = V_Y - V_Y + 1 \]
Substitute 2nd eq to 3rd eq:
\[ V = (V_Y + 100I + 1 - V_Y) + 1 = 100I + 2 \]
or
\[ I = 10^{-2}V - 20 \times 10^{-3} \]

Example

Calculate the I-V relationship of the circuit.

Write a KVL equation for this loop:
\[ +V - 2 - 100I = 0 \]
\[ \Rightarrow I = 10^{-2}V - 20 \times 10^{-3} \]

∴ This circuit has an identical I-V relationship as the previous one ⇒ This circuit is equivalent to the previous circuit.

Equivalent circuit

From the previous slide, ckt (a) and ckt (b) are equivalent. Let’s check:
Substitute ckt (b) into the dash line in (c), we get ckt (d). We can solve this by Ohm’s law:
\[ 2 = I(100 + R) \]
\[ \Rightarrow I = \frac{2}{100 + R} \quad \text{(from subckt (b))} \]
Equivalent circuits example (cont.)

Substitute ckt (c) into the dash line in (c), we get ckt (e). Here we use loop analysis. 2 mesh current $I_1$ and $I$. But $I_1$ is already known. It is equal $10^{-2}$A.

The next loop equation is:

$$100(I-I_1)-1+IR = 0$$

Solve for $I$ yields:

$$I = \frac{2}{100+R} (\text{subckt (a)}) \quad (\text{ccts (d) & (e) result in identical I})$$

∴ The operation of the remainder of a ckt is unaffected when a subcircuit is replaced by its equivalent.

Equivalent circuits

• Two type of equivalent circuits
  – Thévenin equivalent circuits
  – Norton equivalent circuits

Thévenin Equivalent Circuits

General form of Thévenin equivalent is shown in the ckt on right.

$V_T$ is called the Thévenin voltage
$R_T$ is called the Thévenin resistant

By KVL around the loop:

$$V - V_T - R_T I = 0$$

Simplifies into:

$$I = \frac{V}{R_T} - \frac{V_T}{R_T}$$

With $1/R_T$ as the slope in the I-V graph and $-V_T/R_T$ as the "y" intercept c. (Remember $y=mx+c$?)

Thévenin equivalent

In order to find the Thévenin equivalent of a circuit, we need to find $V_T$ and $R_T$. The next question is how to find $V_T$ and $R_T$.

When this circuit is open circuit, $I = 0$.

$$I = \frac{V}{R_T} - \frac{V_T}{R_T}$$

(1)

becomes

$$0 = \frac{V_{OC}}{R_T} - \frac{V_T}{R_T} \quad \Rightarrow \quad V_T = V_{OC}$$

We can compute the $R_T$ by using the $I_{SC}$ (short circuit current):

$$I_{SC} = \frac{0}{R_T} - \frac{V_T}{R_T} \quad \Rightarrow \quad R_T = -\frac{V_T}{I_{SC}} = \frac{V_{OC}}{I_{SC}}$$
**Thévenin equivalent example**

Find the Thévenin equivalent circuit.

We can solve this by 2 steps:
1. Find $V_T$ by finding $V_{OC}$
2. Find $R_T$ by find $I_{SC}$

$V_{OC}$ can be found by inspection (it is a voltage divider):

$$V_T = V_{OC} = V_0 \frac{R_2}{R_1 + R_2} \quad (a)$$

**Thévenin equivalent example cont.**

Next we calculate $I_{SC}$ as shown in circuit on the right.

Apply KCL at node A:

$$\frac{V_0}{R_1} + \frac{0}{R_2} + I_{SC} = 0 \quad \Rightarrow \quad I_{SC} = -\frac{V_0}{R_1} \quad (b)$$

From previous slide,

$$R_T = -\frac{V_{OC}}{I_{SC}} \quad (c)$$

Substitute (b) and (a) into (c)

$$R_T = V_{OC} \left( \frac{R_2}{V_0} \right) = V_0 \left( \frac{R_2}{R_1 + R_2} \right) \left( \frac{R_1}{V_0} \right) = \frac{R_1 R_2}{R_1 + R_2}$$

∴ Circuit © shows a Thévenin equivalent of the original circuit.

**Thévenin equivalent (alternative method)**

There is an alternative method for finding $R_T$:

- Locate all independent voltage & current sources inside the subcircuit whose equivalent is to be found.
- Replace all independent voltage sources by short circuits
- Replace all independent current sources by open circuits
- Compute the resistance between the 2 terminals

**Thévenin equivalent (alternative method) example**

Find the $R_T$ by the alternative method:

- Replace the voltage source by short circuit (current source by open circuit).
- Measure the resistance across the 2 terminals.
- The modified circuit becomes the circuit on the right.

$$R_T = R_1 || R_2$$
Thévenin equivalent (time varying source)

This is identical to the previous example except that it has a time varying source.

The method is the same. The only difference is the source. Instead of \( V_0 \), we use \( V_1 \) as source:

\[
V_1 = 160 \sin(\omega t)
\]

\( V_T = V_{OC} = V_1 \frac{R_2}{R_1 + R_2} \) (time varying source)

\( R_T = V_{OC} \left( \frac{R_1}{V_1} \right) = V_1 \left( \frac{R_2}{R_1 + R_2} \right) \left( \frac{R_1}{V_1} \right) = \frac{R_1 R_2}{R_1 + R_2} \) (time varying source)

In this case, \( RT \) does not depend time, thus, the same.

Norton Equivalent Circuits

Principle of the Norton equivalent circuit is similar to that of the Thevenin equivalent circuit. Consider the general form of Norton equivalent circuit below:

We can find the I-V equation by applying KCL at node A:

\[
I - \frac{V}{R_N} + I_N = 0 \implies I = \frac{V}{R_N} - I_N \quad (1)
\]

Next we need to find \( I_N \) and \( R_N \).

Use similar strategy as in Thevenin equivalent. First short circuit the 2 terminals yields (I becomes \( I_{SC} \), \( V=0 \) in eq. 1):

\[
R_T = \frac{0}{R_N} - I_N \implies I_N = -I_{SC}
\]

Next we open circuit the terminals (I=0, \( V=V_{OC} \) in eq. 1):

\[
0 = \frac{V_{OC}}{R_N} - I_N \implies R_T = -\frac{V_{OC}}{I_{SC}}
\]

Norton equivalent circuits (example)

Find the Norton equivalent ckt of the ckt on the left.

First, find \( I_{SC} \) (the current indicated by an ideal ammeter connected to the terminals A and B.)

Apply KCL at node A in the right ckt:

\[
\frac{V_0 - 0}{R_1} + I_0 + I_{SC} = 0
\]

or

\[
I_{SC} = -\frac{V_0}{R_1} - I_0
\]

Norton equivalent circuits (example cont.)

Next, find \( V_{OC} \) (from ideal voltmeter connected to terminals A&B)

Apply KCL at node A (assume open circuit, I=0) (see circuit c):

\[
\frac{V_0 - V_{OC}}{R_1} + I_0 = 0 \quad \text{or} \quad V_{OC} = I_0 R_1 + V_0
\]

Use \( V_{OC} \) and \( I_{SC} \), we can find \( I_N \) and \( R_N \) (\( I_N = -I_{SC} \), \( R_T = -\frac{V_{OC}}{I_{SC}} \)):

\[
I_N = \frac{V_0}{R_1} + I_0 \quad \text{and} \quad R_N = R_1
\]
Power Calculations

Why?
- To find out how much power is being delivered to (from) some device, e.g. loud speaker.
- Alternative, it may be undesirable to delivered power to some part of the circuit, because of the heat it generates.
- Basic idea is addressed in the first lecture.

How?
Use the Associated Reference Direction convention.

Associated Reference Directions

- It is convenient to define the current through a circuit element as positive when entering the terminal associated with the + reference for voltage

For positive current and positive voltage, positive charge “falls down” a potential “drop” in moving through the circuit element: it absorbs power.

Power Definitions

- $P = VI > 0$ corresponds to the element absorbing power
  - How can a circuit element absorb power?
- By converting electrical energy into heat (resistors in toasters), light (light bulbs), acoustic energy (speakers); by storing energy (charging a battery)
- Negative power - releasing power to the rest of the circuit.

Conservation of Power

- Sum of the power absorbed by all circuit element must be zero.
- Concept: circuit elements are used to model all modes of energy conversion (heat, sound, batteries, voltage generators, etc.)
- Simple example: $I = -2 \text{ mA}$

Power released ($VI < 0$) by the element on the left equals to the power absorbed by the element on the right.
Example of Power Flowing into Current Source

What is the power flow into the current source in the circuit on the right?

Put an imaginary box enclosing the current source and apply the associated reference direction (ARD) to the current source.

(1) Assume V convention on the right, ARD dictates that current has to go into + side of the terminals.

By KVL: \[ V = 100\text{mA} \times (60\Omega + 40\Omega) = 10V \]
By KCL: \[ I = -100\text{mA} \]

The power entering the box (in this case, current source) is:

\[ \text{Power} = VI = 10V \times -100\text{mA} = -1\text{W} \]

∴ 1W is leaving the box.

Example of Power Flowing into Current Source (cont)

(2) Assume V' convention on the right, ARD dictates that current has to go into + side of the terminals.

By KVL: \[ V' = 100\text{mA} \times (60\Omega + 40\Omega) = -10V \]
By KCL: \[ I' = 100\text{mA} \]

The power entering the box (in this case, current source) is:

\[ \text{Power} = V'I' = -10V \times 100\text{mA} = -1\text{W} \]

∴ 1W is leaving the box.

Conclusions: V convention does not affect the results. Both conventions leads to the same conclusion that 1W is leaving the box or current source.

Example of Power Flowing into resistor

Find the power entering the 40 \( \Omega \) resistor as shown in the circuit.

By associate reference direction convention, the V and I are defined.

By inspection \( I = 100\text{mA} \)

\[ V = I \times 40\Omega = 4V \]

\[ P = I \times V = 100\text{mA} \times 4V = 0.4W \]

∴ 0.4W is entering the box (or 40\( \Omega \) resistor).

Using the similar strategy, it is found that 0.6W is entering the 60\( \Omega \) resistor.

Experimental Measurement of Power using a Volt Source and a Ammeter

The box in the right contains an unknown circuit, experimentally it is found that the I-V diagram as shown.

(how? Set the voltage, and measure the current by the ammeter)

<table>
<thead>
<tr>
<th>V (set)</th>
<th>I (measured)</th>
<th>P = V I (compute)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3V</td>
<td>-18mA</td>
<td>+54mW (power entering)</td>
</tr>
<tr>
<td>-2V</td>
<td>-16mA</td>
<td>+32mW</td>
</tr>
<tr>
<td>-1V</td>
<td>-14mA</td>
<td>+14mW</td>
</tr>
<tr>
<td>0</td>
<td>-10mA</td>
<td>0 (no power transfer)</td>
</tr>
<tr>
<td>1V</td>
<td>-2mA</td>
<td>-2mW (power leaving)</td>
</tr>
<tr>
<td>2V</td>
<td>+20mA</td>
<td>+40mW</td>
</tr>
<tr>
<td>3V</td>
<td>+400mA</td>
<td>+1.2W</td>
</tr>
</tbody>
</table>

∴ In the first and the third quadrant of the I-V curve, power is entering.

In the second and the fourth quadrant, power is leaving.
Special cases

If a voltage exists across a resistor $R$,

Power dissipated in the resistor $= VI = V(V/R) = V^2/R$
(Ohm’s law applies in this case)

If a current flow, $I$, through a resistor $R$,

Power dissipated in the resistor $= VI = (IR)I = I^2R$
(Ohm’s law also applies in this case)

Instantaneous Power and Average Power

When $V$ and $I$ are function of time, we write as $v(t)$ and $i(t)$, then instantaneous power entering the box is:

$$P(t) = v(t)i(t)$$
(instantaneous power, power as function of $t$)

This is particularly useful to compute the maximum power received or delivered at any instant.

$$P_{AV} = \frac{1}{T} \int_0^T v(t)i(t)\,dt$$
(time-averaged power)

This is particularly useful for computing the trend, the average power received or delivered.

Does DC current have instantaneous or time-average power?

Example (Time-averaged Power Transferred)

Find the time-averaged power transferred into resistor $R$, if the time varying voltage source $v(t)$ is as shown on the right.

Observation: (1) $v(t)$ is a square wave (max at $V_0$, min at 0)
$i(t)$ is a ______ wave (max at , min at )
Instantaneous power $= v(t)i(t) = \text{ or }$

(2) Period from 0ms to 3ms, or 1ms to 4ms, or 2ms to 5ms.

$$P_{AV} = \frac{1}{0.003 - 0} \left[\int_0^{0.003} v(t)i(t)\,dt + \int_{0.003}^{0.006} \frac{V_0^2}{R}i(t)\,dt + \int_{0.006}^{0.009} 0\,dt\right] = \frac{1}{0.003} \frac{V_0^2}{R} \times (0.001) = \frac{V_0^2}{3R}$$

Multi-terminal Elements

We dealt with circuit element with 2 terminals in the preceding sections.

Here we have 4 terminals

$$P = V_1I_1 + V_2I_2 + V_3I_3 + V_4I_4$$

Notice the reference direction of $I$ is toward the terminals, the terminals are labeled $V_1$, $V_2$, $V_3$, $V_4$.

This can be generalized into any $N$ terminals devices.
Example

Show that the 2 terminals circuit and formula is a special case to our 4 terminals circuit formula.

Is $P=VI$ a special case of $P= V_1I_1+V_2I_2+V_3I_3+V_4I_4$?

Generalize into a 2 terminals circuit:

$P= V_1I_1+V_2I_2$

Comparing the 2 circuits,

$I=I_1$, $I=-I_2$

$P= V_1I_1+V_2I_2$ becomes $P= V_1I-V_2I=(V_1-V_2)I$

But $V=V_1-V_2$ (comparing the 2 circuits)

$P=VI$ which is our original Power formula.

$\therefore P=VI$ is a special case of $P= V_1I_1+V_2I_2+V_3I_3+V_4I_4$ general formula.

Starting a Car

A starting motor of a car typical of a range of 20-40 Amp and 12 volt.

If we take 30Amp and 12 volt.

The motor can be modeled as an equivalent resistance with value $R_m=12/30 = 0.4\Omega$ as shown in the diagram on the right.

Can we replace the bulky car battery by eight 1.5v AA size batteries in series? Why not? They are both 12V.

The battery model $12 \text{ V} \quad \| \quad$ needs to be replaced by a more accurate electrical model.

Battery model

We can model each battery as an ideal 12V voltage source. But this would not help us to understand the difference between eight 1.5V batteries in series and a 12V car battery.

Next, we can model the battery (2 terminal voltage source) as a Thévenin equivalent circuit as shown below.

Both batteries have an identical $V_T$ (open circuit voltage), so what is the difference between the two?

By voltage divider formula:

$V_M = V_T \frac{R_M}{R_T+R_M}$

Since $V_T$ and $R_M$ are the same for both batteries, the difference is in $R_T$.

Battery model (cont)

The typical car battery has a Thévenin resistance of $0.05 \Omega$.

$V_M = 12 \frac{0.4}{0.05+0.4} = 10.67V \quad I = \frac{12}{0.05+0.4} = 26.67A$

The typical AA size battery has a Thévenin resistance of $20 \Omega$.

$V_M = 12 \frac{0.4}{20+0.4} = 0.235V \quad I = \frac{12}{20+0.4} = 0.588A$

That is not sufficient voltage and current to start a car. This is no surprising because you don’t expect AA size batteries to be sufficient to start a car!

Typically, the resistance ($R_T$) is a function of the cross section of the device (in this case, battery). The larger the cross section, the smaller the resistance.
Points To Remember:

- Equivalent circuits are circuits that cannot be distinguished from each other by measurements at their terminals. Often circuit analysis can be simplified if a portion of the circuit is replaced by a simpler equivalent. Two general families of equivalents exist for linear circuits: Thévenin equivalents and Norton equivalents.

- Power flow can be calculated from the expressions $P = VI$ for two-terminal circuit elements and $P = \Sigma V_n I_n$ for multi-terminal circuit element. However, it is essential that the signs of the various voltages and currents be stated correctly.

- If voltage and current vary, the quantity $v(t)i(t)$ is known as the instantaneous power. The time-averaged power is the average over time of the instantaneous power.