

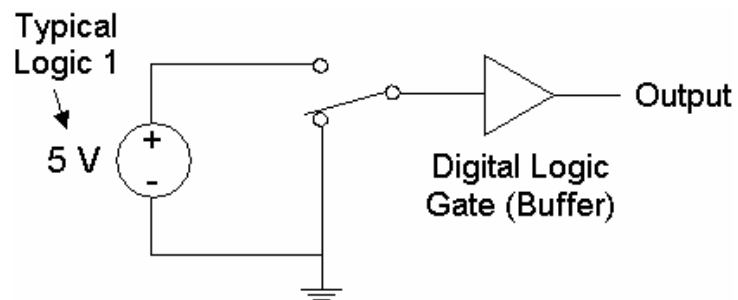
● ● ● | Lecture 11

Today we will learn about **capacitors**

- what they are
- I-V relationship
- their role in the modeling of digital circuits
- how to analyze simple RC circuits

● ● ● | Computation with Voltage

When we perform a sequence of computations using a digital circuit, we switch the input voltages between logic 0 and logic 1.

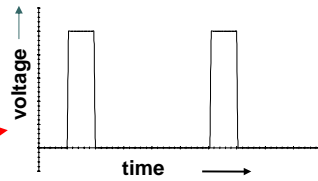


The output of the digital circuit fluctuates between logic 0 and logic 1 as computations are performed.

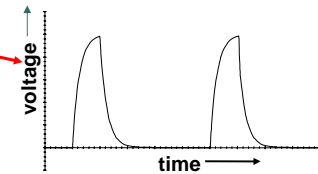
RC Circuits

We compute with pulses

We send beautiful pulses in



But we receive lousy-looking pulses at the output

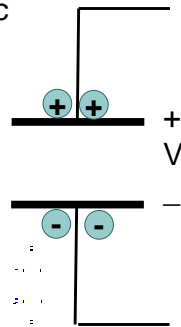


Capacitor charging effects are responsible!

Every node in a circuit has natural capacitance, and it is the charging of these capacitances that limits real circuit performance (speed)

The Capacitor

- The parallel-plate capacitor is made of two metallic plates separated by an insulator such as air.
- Electrons can be ripped off of one plate and transferred to the other, leaving both plates with a nonzero net charge.
- This creates a voltage between the plates.
- The relationship between amount of separated charge and voltage is the capacitor's defining equation:



$Q = C V$ where Q is the amount of charge on the top plate, V is the voltage drop from top to bottom plate, and C is the **capacitance** of the capacitor.

● ● ● | Stored Charge

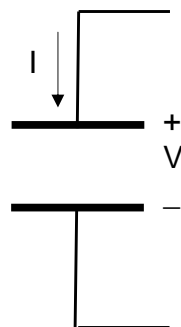
- So far, we have assumed that electrons keep on moving around and around a circuit.
- Current doesn't really "flow through" a capacitor. No electrons can go through the insulator.
- But, we say that current flows "through" a capacitor. What we mean is that positive charge collects on one plate and leaves the other.
- A capacitor stores charge. Theoretically, if we did a KCL surface around one plate, KCL could fail. But we don't do that.
- When a capacitor stores charge, it has nonzero voltage. In this case, we say the capacitor is "charged". A capacitor with zero voltage has no charge differential, and we say it is "discharged".

● ● ● | I-V Relationship

- We said that $Q = C V$

where Q is the amount of charge on the top plate, V is the voltage drop from top to bottom plate, and C is the capacitance of the capacitor.

- Taking the time derivative of both sides,



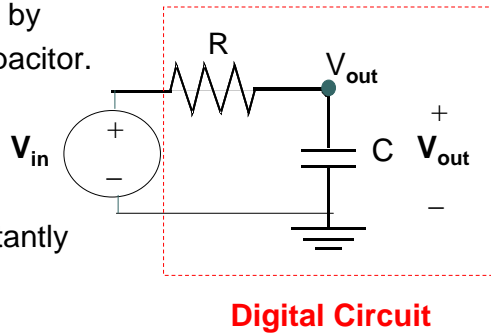
- Remembering the definition of current, $I = C \frac{dV}{dt}$

RC Circuit Model

The capacitor is used to model the response of a digital circuit to a new voltage input:

The digital circuit is modeled by a resistor in series with a capacitor.

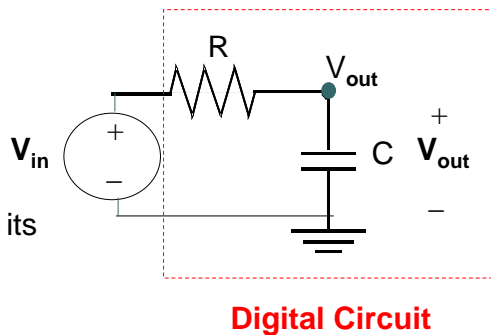
The capacitor cannot change its voltage instantly, as charges can't teleport instantly to the other plate.



RC Circuit Model

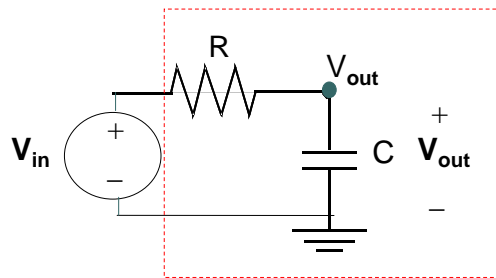
Every digital circuit has natural resistance and capacitance. In real life, the resistance and capacitance can be estimated using characteristics of the materials used and the layout of the physical device.

The value of R and C for a digital circuit determine how long it will take the capacitor to change its voltage—the gate delay.

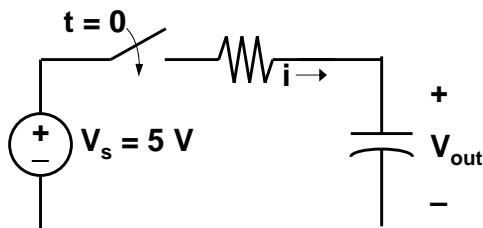


RC Circuit Model

With the digital context in mind, V_{in} will usually be a time-varying voltage that switches instantaneously between logic 1 voltage and logic 0 voltage.

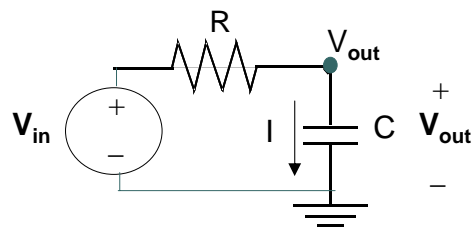


We often represent this switching voltage with a switch in the circuit diagram.



Analysis of RC Circuit

By KVL,

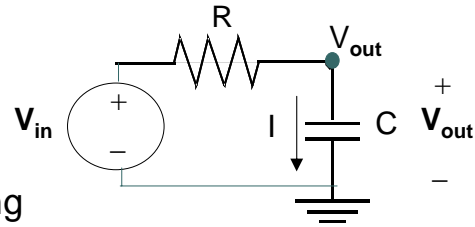


Using the capacitor I-V relationship,

We have a first-order linear nonhomogeneous differential equation, with characteristic equation root $-1/(RC)$.

● ● ● | Analysis of RC Circuit

- What does that mean?
- One could solve the differential equation using Math 54 techniques to get

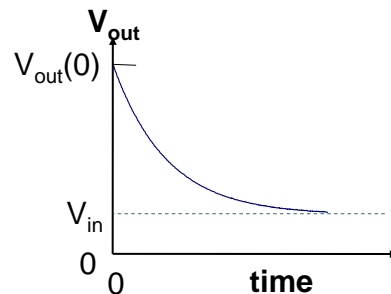
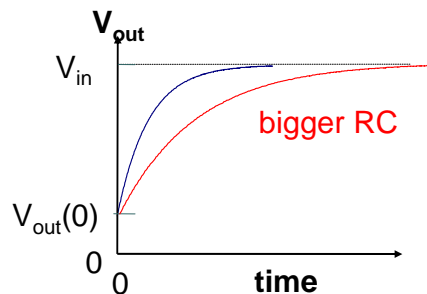


$$V_{\text{out}}(t) = V_{\text{in}} + (V_{\text{out}}(0) - V_{\text{in}})e^{-t/(RC)}$$

● ● ● | Insight

$$V_{\text{out}}(t) = V_{\text{in}} + (V_{\text{out}}(0) - V_{\text{in}})e^{-t/(RC)}$$

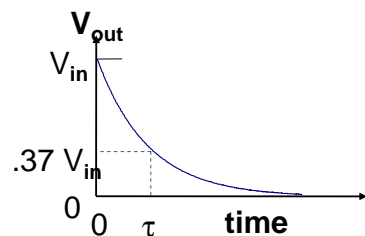
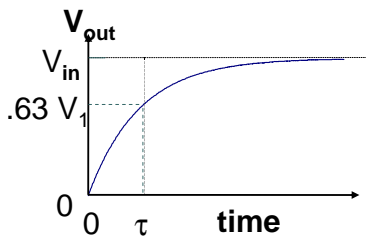
- $V_{\text{out}}(t)$ starts at $V_{\text{out}}(0)$ and goes to V_{in} asymptotically.
- The difference between the two values decays exponentially.
- The rate of convergence depends on RC . The bigger RC is, the slower the convergence.



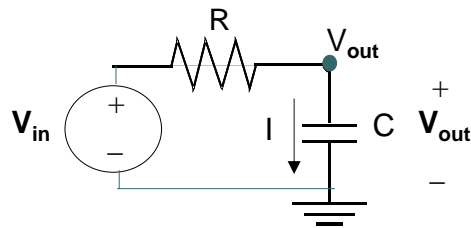
Time Constant

$$V_{\text{out}}(t) = V_{\text{in}} + (V_{\text{out}}(0) - V_{\text{in}})e^{-t/(RC)}$$

- The value RC is called the **time constant**.
- After 1 time constant has passed ($t = RC$), the above works out to:
- So after 1 time constant, $V_{\text{out}}(t)$ has completed 63% of its transition, with 37% left to go.
- After 2 time constants, only 0.37^2 left to go.



Transient vs. Steady-State



- When V_{in} does not match up with V_{out} , due to an abrupt change in V_{in} for example, V_{out} will begin its **transient period** where it exponentially decays to the value of V_{in} .
- After a while, V_{out} will be close to V_{in} and be nearly constant. We call this **steady-state**.
- In steady state, the current through the capacitor is (approx) zero. **The capacitor behaves like an open circuit in steady-state.**
- Why? $I = C \, dV_{\text{out}}/dt$, and V_{out} is constant in steady-state.

● ● ● | General RC Solution

- Every current or voltage (except the source voltage) in an RC circuit has the following form:

$$x(t) = x_f + \left(x(t_0^+) - x_f \right) e^{-t_0 / (RC)}$$

- x represents any current or voltage
- t_0 is the time when the source voltage switches
- x_f is the final (asymptotic) value of the current or voltage

All we need to do is find these values and plug in to solve for any current or voltage in an RC circuit.

● ● ● | Solving the RC Circuit

We need the following three ingredients to fill in our equation for any current or voltage:

- $x(t_0^+)$ This is the current or voltage of interest just after the voltage source switches. It is the starting point of our transition, the initial value.
- x_f This is the value that the current or voltage approaches as t goes to infinity. It is called the final value.
- RC This is the time constant. It determines how fast the current or voltage transitions between initial and final value.

● ● ● | Finding the Initial Condition

To find $x(t_0^+)$, the current or voltage just after the switch, we use the following essential fact:

Capacitor voltage is continuous; it cannot jump when a switch occurs.

So we can **find the capacitor voltage** $V_C(t_0^+)$ by finding $V_C(t_0^-)$, the voltage **before switching**.

We can **assume the capacitor was in steady-state** before switching. The capacitor acts like an open circuit in this case, and it's not too hard to find the **voltage over this open circuit**.

We can then find $x(t_0^+)$ using $V_C(t_0^+)$ using KVL or the capacitor I-V relationship. These laws hold for every instant in time.

● ● ● | Finding the Final Value

To find x_f , the asymptotic final value, we assume that the circuit will be in **steady-state** as t goes to infinity.

So we assume that the capacitor is acting like an open circuit. We then find the value of current or voltage we are looking for using this open-circuit assumption.

Here, we use the circuit **after switching** along with the open-circuit assumption.

When we found the initial value, we applied the open-circuit assumption to the circuit before switching, and found the capacitor voltage which would be preserved through the switch.

● ● ● | Finding the Time Constant

It seems easy to find the time constant: it equals RC .

But what if there is more than one resistor or capacitor?

R is the Thevenin equivalent resistance with respect to the capacitor terminals.

Remove the capacitor and find R_{TH} . It might help to turn off the voltage source. Use the circuit **after switching**.

We will discuss how to combine capacitors in series and in parallel in the next lecture.

● ● ● | Alternative Method

Instead of finding these three ingredients for the generic current or voltage x , we can

- Find the initial and final capacitor voltage (it's easier)
- Find the time constant (it's the same for everything)
- Form the capacitor voltage equation $V_C(t)$
- Use KVL or the I-V relationship to find $x(t)$