1. Convexity of Sets

Definition. A set \( C \) is convex if and only if the line segment between any two points in \( C \) lies in \( C \):

\[
C \text{ is convex } \iff \forall \vec{x}_1, \vec{x}_2 \in C, \forall \theta \in [0, 1], \theta \vec{x}_1 + (1 - \theta) \vec{x}_2 \in C
\]

(a) Show that the intersection of convex sets is convex:

\[
C_1, C_2 \text{ are convex } \implies C = C_1 \cap C_2 \text{ is convex}
\]

(b) Show that the following sets are convex:

i. [Optional] A vector subspace of \( \mathbb{R}^n \)

ii. [Optional] A hyperplane, \( \mathcal{L} = \{ \vec{x} \mid \vec{a}^\top \vec{x} = b \} \).

iii. A halfspace, \( \mathcal{H} = \{ \vec{x} \mid \vec{a}^\top \vec{x} \leq b \} \).

Definition. A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is affine if it is the sum of a linear function and a constant,

\[
f(\vec{x}) = A\vec{x} + \vec{b},
\]

for \( A \in \mathbb{R}^{m \times n} \) and \( \vec{b} \in \mathbb{R}^m \).

(c) [Optional] Conservation of convexity through affine transformation. Prove that if \( S \subseteq \mathbb{R}^n \) is convex, then the image of \( S \) under an affine function \( f \),

\[
f(S) = \{ f(\vec{x}) \mid \vec{x} \in S \},
\]

is convex.

2. Convexity of Functions

Definition. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is convex if \( \text{dom}(f) \) is a convex set and if for all \( \vec{x}, \vec{y} \in \text{dom}(f) \) and \( \theta \in [0, 1] \), we have,

\[
f(\theta \vec{x} + (1 - \theta) \vec{y}) \leq \theta f(\vec{x}) + (1 - \theta) f(\vec{y}). \tag{1}
\]

The function \( f \) is strictly convex if the inequality is strict.

Definition. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is concave if \( \text{dom}(f) \) is a convex set and if for all \( \vec{x}, \vec{y} \in \text{dom}(f) \) and \( \theta \) with \( 0 \leq \theta \leq 1 \), we have,

\[
f(\theta \vec{x} + (1 - \theta) \vec{y}) \geq \theta f(\vec{x}) + (1 - \theta) f(\vec{y}).
\]

The function \( f \) is strictly concave if the inequality is strict.

Property. A function \( f \) is concave if and only if \(-f\) is convex. An affine function is both convex and concave.
Property: Jensen’s inequality. The inequality in Equation (1) is known as Jensen’s Inequality. This can be extended to convex combinations of more than one point. If \( f \) is convex, and \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k \in \text{dom}(f) \), and \( \theta_1, \theta_2, \ldots, \theta_k \geq 0 \) with \( \sum_{i=1}^{k} \theta_i = 1 \) then,

\[
f(\theta_1 \bar{x}_1 + \theta_2 \bar{x}_2 + \cdots + \theta_k \bar{x}_k) \leq \theta_1 f(\bar{x}_1) + \theta_2 f(\bar{x}_2) + \cdots + \theta_k f(\bar{x}_k).
\]

Property: First order condition. Suppose \( f \) is differentiable. Then \( f \) is convex if and only if \( \text{dom}(f) \) is convex and

\[
f(\bar{y}) \geq f(\bar{x}) + \nabla f(\bar{x})^T (\bar{y} - \bar{x}),
\]

for all \( \bar{x}, \bar{y} \in \text{dom}(f) \).

Property: Second order condition. Suppose \( f \) is twice differentiable. Then \( f \) is convex if and only if, \( \text{dom}(f) \) is convex and the Hessian of \( f \), \( \nabla^2 f(\bar{x}) \), is positive semi-definite for all \( \bar{x} \in \text{dom}(f) \).

(a) Under what condition on \( A \in \mathbb{R}^{n \times n} \), where \( A \) is symmetric, is the function \( f : \bar{x} \rightarrow \bar{x}^T A \bar{x} \) convex?

(b) [Optional] Restriction to a line. Show that a function \( f \) is convex if and only if for all \( \bar{x} \in \text{dom}(f) \) and all \( \bar{v} \), the function \( g : \text{dom}(g) \rightarrow \mathbb{R} \) given by \( g(t) = f(\bar{x} + t\bar{v}) \) is convex for \( \text{dom}(g) = \{ t \in \mathbb{R} \mid \bar{x} + t\bar{v} \in \text{dom}(f) \} \).

(c) [Optional] Non-negative weighted sum. Show that the non-negative weighted sum of convex functions is convex: i.e. if \( f_1, \ldots, f_n \) are \( n \) convex functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) and \( w_1, \ldots, w_n \in \mathbb{R}_+ \) are \( n \) positive scalars, then the function:

\[
f = \sum_{i=1}^{n} w_i f_i
\]

is convex. To make the question easier, you can assume that the functions \( f_1, \ldots, f_n \) are twice-differentiable.

(d) [Optional] Point-wise maximum Show that if \( f_1 \) and \( f_2 \) are convex functions then their pointwise maximum \( f \), defined by

\[
f(\bar{x}) = \max(f_1(\bar{x}), f_2(\bar{x})),
\]

with \( \text{dom}(f) = \text{dom}(f_1) \cap \text{dom}(f_2) \), is also convex.

(e) Show that a piece-wise linear function that can be written as,

\[
f(\bar{x}) = \max(\bar{a}_1^T \bar{x} + \bar{b}_1, \bar{a}_2^T \bar{x} + \bar{b}_2, \ldots, \bar{a}_m^T \bar{x} + \bar{b}_m),
\]

is convex.

3. Disproving convexity: Finding counter-examples

Though we spend a lot of time in this course learning how to prove convexity of sets and functions, in practical scenarios we may not have a mathematical representation of a set/function and so it is not possible to prove convexity. Instead, we may be able to represent this set/function in terms of a query \( Q(\bar{x}) \) that returns some information about the element \( \bar{x} \) in relation to the set/function.
For example, instead representing the set \( S = \{ \vec{x} \mid \text{some condition on } \vec{x} \} \) we only have \( Q(\vec{x}) \) which returns whether or not \( \vec{x} \in S \).

In these cases we can **disprove** convexity by showing that one or more of the properties of convex sets/functions are violated by finding counterexamples. In this problem we will see how we can disprove convexity for sets/functions given limited information that can be accessed via certain types of queries.

(a) **Disproving convexity of set \( S \) (Proving non-convexity of set \( S \))**

Assume that we know that the set lies within some \( D \).

Query: \( Q(\vec{x}) \): For \( \vec{x} \in D \) that returns True if \( \vec{x} \in S \) and False if \( \vec{x} \notin S \). How can you use \( Q \) to check/disprove convexity of \( S \)?

(b) **Disproving convexity of function \( f \) (Proving non-convexity of function \( f \)).**

Assume that we know \( \text{dom}(f) \), denoted as \( D \) and that \( D \) is convex.

i. Query: \( G(\vec{x}) \): For \( \vec{x} \in D \), returns function value \( f(\vec{x}) \). How can you use \( G \) to check/disprove convexity of \( f \)?

ii. Query: \( H(\vec{x}) \): For \( \vec{x} \in D \), returns \( f(\vec{x}) \) and \( \nabla f(\vec{x}) \). (Here we assume that \( f \) is differentiable). How can you use \( H \) to check/disprove convexity of \( f \)?