1 Eigenvalues of an upper triangular matrix

We will show, without determinants, that the eigenvalues of an upper triangular matrix are its diagonal entries.

Throughout this problem, $A \in \mathbb{C}^{n \times n}$ will be a general square matrix, and $a_{ij}$ will denote its coordinate at row $i$, column $j$.

An eigenvalue of $A$ is a scalar $\lambda \in \mathbb{C}$ such that $A - \lambda I$ does not have full rank. Observe this condition is equivalent to the existence of a nonzero vector $\tilde{v} \in \mathbb{C}^n$ such that $A\tilde{v} = \lambda \tilde{v}$.

a) As $A$ is upper triangular, it has the following form:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \ldots & a_{2,n} \\ 0 & 0 & a_{3,3} & \ldots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{n,n} \end{bmatrix}.$$ \hspace{1cm} (1)

Show that if $a_{k,k}$ is any diagonal value of $A$, $A - a_{k,k} I$ does not have full rank. (Hint: how do you check the rank of a matrix by row reduction?)

Solution

We will prove this by row reduction. Subtraction of $a_{k,k} I$ clears diagonal entries of $A$ that were equal to $a_{k,k}$. Let row $i^*$ be the last row of $A$ whose diagonal entry was cancelled by subtracting $a_{k,k}$. The lower right square submatrix of order $(n - i^* + 1)$ has only zeros in its leftmost column. As such, it cannot have full rank.

Therefore, among the rightmost $(n - i^* + 1)$ columns of $(A - \lambda I)$ there are strictly fewer than $(n - i^* + 1)$ pivots.

But these columns must have $(n - i^* + 1)$ pivots in order for $A$ to have full rank. Therefore $A$ cannot have full rank.

b) Show that if $A - \lambda I$ does not have full rank, $\lambda$ is equal to a diagonal value of $A$.

Solution

If $\lambda$ is not a diagonal value of $A$, then $A - \lambda I$ is in row-echelon form with a pivot in each column. Then $A$ has full rank.

2 SVD (40 points)

a) Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $\tilde{x} \in \mathbb{R}^n$ be a nonzero vector. Prove that $\|A\tilde{x}\| \geq \sigma_{\min} \|\tilde{x}\|$.
Solution

Without loss of generality, let \( \mathbf{x} \) be a unit vector. If \( A = U \Sigma V^T \),

\[
A \mathbf{x} = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \mathbf{x}
\]

(2)

\[
= \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \left( \mathbf{v}_i^T \mathbf{x} \right)
\]

(3)

\[
\|A \mathbf{x}\|^2 = \left\| \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \left( \mathbf{v}_i^T \mathbf{x} \right) \right\|^2
\]

(4)

By the Pythagorean theorem,

\[
\sum_{i=1}^{n} \sigma_i^2 \left\| \mathbf{u}_i \left( \mathbf{v}_i^T \mathbf{x} \right) \right\|^2
\]

(5)

A minimum is achieved at \( \mathbf{x} = \mathbf{v}_{\text{min}} \) so that only one term of the sum remains.

\[
= \sigma_{\text{min}}^2
\]

(6)

b) Let \( A \in \mathbb{R}^{2 \times 2} \) and \( \mathbf{x} = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix} \), \( \|\mathbf{x}\| = 1 \). Now let \( \mathbf{y} = A \mathbf{x} \). Below is the plot of \( \|\mathbf{y}\| \) vs \( \theta \).

\[A\] has the SVD \( U \Sigma V^T \). Either specify what the matrices \( U \), \( \Sigma \), and \( V \) are; or state they cannot be determined from the information given.
Solution

We know that $\sigma_2 \leq ||A\vec{x}|| \leq \sigma_1$, so from the above graph we can see that $\sigma_1 = 5$ and $\sigma_2 = 1$.

These occur for $\vec{x} = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \vec{v}_1$ and $\vec{x} = \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \vec{v}_2$ respectively. Since we observe $||\vec{y}||$, $\vec{y}$ can be arbitrarily rotated by $U$, so we cannot deduce a unique $U$.

c) Let $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times N}$ be full rank matrices and let $\vec{x} \in \mathbb{R}^N$ have $||\vec{x}|| = 1$. Let $\vec{y} = AB\vec{x}$. Find a upper bound for $||\vec{y}||$ in terms of the singular values of $A$ and $B$. Explain your answer.

Solution

\[
\begin{align*}
||B\vec{x}|| &\leq \sigma_{\text{max}} \{A\} \\
\begin{bmatrix} A & \vec{x} \\ \vec{x}^T & ||\vec{x}|| \end{bmatrix} &\leq \sigma_{\text{max}} \{A\}
\end{align*}
\]

Where $\sigma_{\text{max}} \{M\}$, for some matrix $M$, is the largest singular value of $M$.

If $\vec{x} = \vec{v}_1 \{B\}$ and $B\vec{x} = \vec{v}_1 \{A\}$, then the output is maximal, with

\[
||AB\vec{x}|| = \sigma_{\text{max}} \{A\} \cdot \sigma_{\text{max}} \{B\}
\]

3 Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Arcak told him that if his system has complex eigenvalues $\lambda_{1,2} = \nu \pm j\omega$, then his altitude would indeed oscillate with frequency $\omega$ about the steady state value, 1 km, and that the time trace of his altitude would be tangent to the curves $1 + e^{\nu t}$ and $1 - e^{\nu t}$ near its maxima and minima respectively.
a) Find the real part $v$ and the imaginary part $\omega$ from the altitude plot.

**Solution**

Solving $1 + e^{5v} = 1.4843$ gives us $v = -0.1450 \frac{1}{\text{min}}$. Then, comparing the maxima that are separated by an interval of 10 minutes gives $\omega = \frac{\frac{\pi}{10}}{10} = 0.0628 \frac{\text{rad}}{\text{min}}$.

If you solved in units of $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{s}$, then $v = -0.0024 \frac{1}{s}$ and $\omega = 0.0105 \frac{\text{rad}}{s}$.

b) Let the dynamical model for the altitude be

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and $a_1$ and $a_2$ are constants. Using your answer to part (a), find what $a_1$ and $a_2$ are.

**Solution**

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$ are given by $0 = \lambda^2 - a_2 \lambda - a_1$, or equivalently,

$$\lambda = \frac{a_2 \pm \sqrt{a_2^2 + 4a_1}}{2} = v \mp j\omega.$$

Solving for $a_1$ and $a_2$ (using the $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{min}}$ values of $v$ and $\omega$), we get

$$a_2 = 2v = -0.2900 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -0.4158.$$
If you solved using the $\frac{1}{s}$ and $\frac{rad}{s}$ values of $v$ and $\omega$, then

\[ a_2 = 2v = -0.0048 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -1.16 \cdot 10^{-4}. \]

c) Otto can change $a_2$ by turning a knob. Tell him what value he should pick so that he has a "critically damped" ascent with two real negative eigenvalues at the same location.

**Solution**

To get two real identical eigenvalues, Otto should choose $a_2$ to make $a_2^2 + 4a_1 = 0$. This means that $a_2 = \pm 2\sqrt{-a_1}$. Since $a_2$ must be negative for the system to be stable, we only look at the negative root.

Solving with the $a_1$ derived from the $\frac{1}{\text{min}}$ and $\frac{rad}{\text{min}}$ values of $v$ and $\omega$, he should tune his knob to

\[ a_2 = -2\sqrt{-a_1} = -2\sqrt{0.4158} = -1.2897. \]

If you solved using $a_1$ derived from the $\frac{1}{\text{min}}$ and $\frac{rad}{\text{min}}$ values of $v$ and $\omega$, then you get

\[ a_2 = -2\sqrt{-a_1} = -2\sqrt{1.16 \cdot 10^{-4}} = -0.0215. \]

4 Balance — linearizing a vector system

Justin is working on a small jumping robot named Salto. Salto can bounce around on the ground, but Justin would like Salto to balance on its toe and stand still. In this problem, we’ll work on systems that could help Salto balance on its toe using its reaction wheel tail.
Standing on the ground, Salto’s dynamics in the $x$-$z$ plane (called the sagittal plane in biology) look like an inverted pendulum with a flywheel on the end,

\[
(I_1 + (m_1 + m_2)t^2) \frac{d^2 \theta_1(t)}{dt^2} = -K_1 u(t) + (m_1 + m_2)g \sin(\theta_1(t))
\]

\[
I_2 \frac{d^2 \theta_2(t)}{dt^2} = K_1 u(t),
\]

where $\theta_1(t)$ is the angle of the robot’s body relative to the ground at time $t$ ($\theta_1 = 0$ rad means the body is exactly vertical), $\frac{d\theta_1(t)}{dt}$ is the robot body’s angular velocity, $\frac{d\theta_2(t)}{dt}$ is the angular velocity of the reaction wheel tail, and $u(t)$ is the current input to the tail motor. $m_1, m_2, I_1, I_2, l, K_1$ are positive constants representing system parameters (masses and angular momentums of the body and tail, leg length, and motor torque constant, respectively) and $g = 9.81 \frac{m}{s^2}$ is the acceleration due to gravity.

Numerically substituting Salto’s physical parameters, the differential equations become:

\[
0.001 \frac{d^2 \theta_1(t)}{dt^2} = -0.025 u(t) + 0.1 \sin(\theta_1(t))
\]

\[
5(10^{-5}) \frac{d^2 \theta_2(t)}{dt^2} = 0.025 u(t)
\]
a) Using the state vector \[
\begin{bmatrix}
\theta_1 \\
\frac{d\theta_1}{dt} \\
\frac{d\theta_2}{dt} \\
\frac{d^2\theta_2}{dt^2}
\end{bmatrix}
\]
and input \(u\), linearize the system about the point \(\bar{\mathbf{x}}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\) with nominal input \(u^* = 0\). Write the linearized equation as \(\frac{d}{dt}\bar{\mathbf{x}} = A\bar{\mathbf{x}} + Bu\). Write out the matrices with the physical numerical values.

Note: Since the tail is like a wheel, we care only about the tail’s angular velocity \(\frac{d\theta_2}{dt}\) and not its angle \(\theta_2(t)\). This is why \(\theta_2(t)\) is not a state.

Hint: The \(\sin\) is the only nonlinearity that you have to deal with here.

Solution

With state vector \(\bar{\mathbf{x}} = \begin{bmatrix} \theta_1(t) \\ \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \\ \frac{d^2\theta_2}{dt^2} \end{bmatrix}\), we want to write the dynamics in the form

\[
\begin{bmatrix}
\frac{d\theta_1}{dt} \\
\frac{d^2\theta_1}{dt^2} \\
\frac{d\theta_2}{dt} \\
\frac{d^2\theta_2}{dt^2}
\end{bmatrix} = A
\begin{bmatrix}
\theta_1(t) \\
\frac{d\theta_1}{dt} \\
\frac{d\theta_2}{dt} \\
\frac{d^2\theta_2}{dt^2}
\end{bmatrix} + Bu(t).
\]

From the problem statement, we know that

\[
\begin{align*}
\frac{d\theta_1}{dt} &= \frac{d\theta_1}{dt} \\
\frac{d^2\theta_1}{dt^2} &= 100\sin(\theta_1(t)) - 25u(t) \\
\frac{d^2\theta_2}{dt^2} &= 500u(t).
\end{align*}
\]

The only nonlinear component is the \(\sin(\theta_1(t))\) in the \(\frac{d^2\theta_1}{dt^2} = 100\sin(\theta_1(t)) - 25u(t)\).
25u(t) equation, which we want to linearize about the point 
\[ x^* = \begin{bmatrix} \theta_1(t)^* \\ \frac{d\theta_1(t)}{dt} \\ \frac{d\theta_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \]

Let’s rename this equation as
\[ f(\theta_1(t)) = \frac{d^2\theta_1(t)}{dt^2} = 100\sin(\theta_1(t)) - 25u(t) \]
for notational simplicity.

We are finding the tangent line through our operating point \( \bar{x}^* \) in order to approximate this function. (When we approximate a function using a line - that is, we linearize it - we can use matrices to represent the function.) We note that the only relevant operating point in this equation is \( \theta_1(t)^* \), since this is the only state variable in the equation under consideration.

We find the intersect of the line (which is the function evaluated at the operating point, \( f(\theta_1(t)^*) \)) and the slope of the line (the derivative of the function at the operating point, \( f'(\theta_1(t)^*) \)) in order to write this linear equation around the operating point. Our linearized equation will then be given by
\[ f(\theta_1(t)) = f'(\theta_1(t)^*)\theta_1(t) + f(\theta_1(t)^*). \]

Accordingly,
\[ f(\theta_1(t)^* = 0) = 100\sin(0) - 25u(t) = -25u(t) \]
\[ f(\theta_1(t)^* = 0)' = \frac{\partial}{\partial \theta_1(t)} (100\sin(\theta_1(t)) - 25u(t))|_{\theta_1(t)^*} \]
\[ = 100\cos(\theta_1(t))|_{\theta_1(t)^*} \]
\[ = 100 \cos(0) \]
\[ = 100 \]

Therefore, the linearized equation is given by
\[ f(\theta_1(t)) = \frac{d^2\theta_1(t)}{dt^2} = 100\theta_1(t) - 25u(t). \]

Now that all of our equations are linear, we can represent the dynamics
with a matrix. Plugging into the matrix form gives:

\[
\begin{bmatrix}
\frac{d\theta_1(t)}{dt} \\
\frac{d^2\theta_1(t)}{dt^2} \\
\frac{d\theta_2(t)}{dt} \\
\frac{d^2\theta_2(t)}{dt^2}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
100 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\frac{d\theta_1(t)}{dt} \\
\frac{d\theta_2(t)}{dt}
\end{bmatrix} +
\begin{bmatrix}
0 \\
-25 \\
500
\end{bmatrix} u(t)
\]

It is just that single entry in the matrix with the \( \sin \) that needs to be approximated for small \( \theta_1 \), and it is clear that there, \( \frac{\sin(\theta_1)}{\theta_1} \approx 1 \).

b) Your linearized system should have at least one eigenvalue that corresponds to a growing exponential. If we just do the formal test for controllability by checking the \((A, B)\) pair for the linearized system, does it indicate that we could place the closed-loop eigenvalues wherever we want for the linearized system?

**Solution**

\[
C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} =
\begin{bmatrix}
0 & -25 & 0 \\
-25 & 0 & -2500 \\
500 & 0 & 0
\end{bmatrix}
\]

which is full rank, so the system is fully controllable. Because of the proof of CCF, this tells us that we can put the eigenvalues wherever we want for the linearized system. This is because even in continuous time, the closed-loop dynamics are given by \( A - BK \) if the control is \( u(t) = -Kx(t) \). The CCF proof told us that we can set the characteristic polynomial (and hence the eigenvalues) of \( A - BK \) to be whatever we want.

c) Using state feedback, Justin has selected the control gains \( K = \begin{bmatrix} 20 & 5 & 0.01 \end{bmatrix} \) for his input \( u = K\ddot{x} \). What are the eigenvalues of the closed loop dynamics for the given \( K \)?

Feel free to use numpy.
Solution

With this closed loop feedback, our dynamics become:

\[
\begin{bmatrix}
\frac{d\theta_1(t)}{dt} \\
\frac{d^2\theta_1(t)}{dt^2} \\
\frac{d^2\theta_2(t)}{dt^2}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
100 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{d\theta_1(t)}{dt} \\
\frac{d\theta_2(t)}{dt} \\
\frac{d^2\theta_2(t)}{dt^2}
\end{bmatrix} +
\begin{bmatrix}
0 \\
-25 \\
500
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\frac{d\theta_1(t)}{dt} \\
\frac{d\theta_2(t)}{dt}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 \\
-500 & -125 & -0.25 \\
10000 & 2500 & 5
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\frac{d\theta_1(t)}{dt} \\
\frac{d\theta_2(t)}{dt}
\end{bmatrix}
\]

Using numpy to solve, we find the eigenvalues are \( \lambda = -116.6 \) and \( \lambda = -1.697 \pm 1.187i \).