Recap: Inference Example

- Find \(P(W|F=\text{bad})\)
  - Restrict all factors
  - No hidden vars to eliminate (this time!)
  - Just join and normalize

\[
\begin{array}{c|c|c}
W & P(W) & \text{Weather} \\
--- & --- & --- \\
\text{sun} & 0.7 & \text{sun} \\
\text{rain} & 0.3 & \text{rain} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
F & P(F|\text{work}) & \text{Forecast} & P(F|\text{rain}) & P(F|\text{sun}) \\
--- & --- & --- & --- & --- \\
\text{bad} & 0.8 & \text{bad} & 0.2 & \text{bad} \\
\text{good} & 0.1 & \text{good} & 0.3 & \text{good} \\
\end{array}
\]

\[
f_{W|F}(W) = f_W(W) \times f_{W|F}(W)
\]

\[
P(W|F=\text{bad}) = \frac{f_{W|F}(W)}{\sum_W f_{W|F}(W)}
\]

Decision Networks

- MEU: choose the action which maximizes the expected utility given the evidence
- Can directly operationalize this with decision diagrams
  - Bayes nets with nodes for utility and actions
  - Lets us calculate the expected utility for each action
- New node types:
  - Chance nodes (just like BNs)
  - Actions (rectangles, must be parents, act as observed evidence)
  - Utilities (depend on action and chance nodes)

Example: Decision Networks

Umbrella = leave

\[
\begin{align*}
\text{EU(leave)} &= \sum_w P(w)U(\text{leave}, w) \\
&= 0.7 \cdot 100 + 0.3 \cdot 0 = 70
\end{align*}
\]

Optimal decision = leave

\[
\text{MEU(\text{leave}) = max } \text{EU(\text{leave})} = 70
\]

Umbrella = take

\[
\begin{align*}
\text{EU(take)} &= \sum_w P(w)U(\text{take}, w) \\
&= 0.7 \cdot 20 + 0.3 \cdot 70 = 37
\end{align*}
\]

Optimal decision = take

\[
\text{MEU(\text{take}) = max } \text{EU(\text{take})} = 70
\]
**Value of Information**

- Idea: compute value of acquiring each possible piece of evidence
  - Can be done directly from decision network
- Example: buying oil drilling rights
  - Two blocks A and B, exactly one has oil, worth k
  - Prior probabilities 0.5 each, mutually exclusive
  - Current price of each block is k/2
  - Probe gives accurate survey of A. Fair price?
- Solution: compute value of information
  - expected value of best action given the information minus expected value of best action without information
  - Survey may say “oil in A” or “no oil in A,” prob 0.5 each
  
  \[
  \text{value of information} = [0.5 \times \text{value of "buy A" given "oil in A"}] + [0.5 \times \text{value of "buy B" given "no oil in A"}] - 0
  \]
  
  \[= [0.5 \times k/2] + [0.5 \times k/2] - 0 = k/2\]

**VPI Example**

- MEU with no evidence
  
  \[\text{MEU}(a) = \max_a \mu_a\]

- MEU if forecast is bad
  
  \[\text{MEU}(F = \text{bad}) = \max_a \mu_a(\text{bad}) = 52\]

- MEU if forecast is good
  
  \[\text{MEU}(F = \text{good}) = \max_a \mu_a(\text{good}) = 96\]

- Forecast distribution

<table>
<thead>
<tr>
<th>Forecast</th>
<th>umbrella</th>
<th>umbrella</th>
</tr>
</thead>
<tbody>
<tr>
<td>good</td>
<td>0.3</td>
<td>0.7</td>
</tr>
<tr>
<td>bad</td>
<td>0.7</td>
<td>0.3</td>
</tr>
</tbody>
</table>

  \[E' = \sum_{c'} P(c'|F) (\text{MEU}(c, c') - \text{MEU}(c))\]

- Value of information

**Value of Information**

- Current evidence E=a, utility depends on S=s

  \[\text{MEU}(c) = \sum_s P(s|a) U(s, a)\]

  \[\text{MEU}(c, c') = \sum_s P(s|a) U(s, a)\]

  \[\text{VPI}(E) = \sum_s P(c'|s) (\text{MEU}(c, c') - \text{MEU}(c))\]

**VPI Properties**

- Nonnegative in expectation
  
  \[\forall j, c : \text{VPI}(E) \geq 0\]

- Nonadditive ---consider, e.g., obtaining E twice

  \[\text{VPI}(E_1, E_2) \neq \text{VPI}(E_1) + \text{VPI}(E_2)\]

- Order-independent

  \[\text{VPI}(E_1, E_2) = \text{VPI}(E_2, E_1)\]

**VPI Scenarios**

- Imagine actions 1 and 2, for which U_1 > U_2
  - How much will information about E_i be worth?

**Reasoning over Time**

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring

- Need to introduce time into our models
- More general: dynamic Bayes’ nets
Markov Models

A Markov model is a chain-structured BN
- Each node is identically distributed (stationarity)
- Value of X at a given time is called the state
- As a BN:

\[
P(X_1) \quad P(X|X_{-1})
\]

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial probs)

Conditional Independence

- Basic conditional independence:
  - Past and future independent of the present
  - Each time step only depends on the previous
    - This is called the (first order) Markov property
  - Note that the chain is just a (growing) BN
    - We can always use generic BN reasoning on it (if we truncate the chain)

Example: Markov Chain

Weather:
- States: X = {rain, sun}
- Transitions:

\[
\begin{align*}
P(X_2 = sun) &= P(X_2 = sun|X_1 = sun)P(X_1 = sun) + P(X_2 = sun|X_1 = rain)P(X_1 = rain) \\
&= 0.9 \cdot 1.0 + 0.1 \cdot 0.0 = 0.9
\end{align*}
\]

Mini-Forward Algorithm

- Question: probability of being in state x at time t?
- Slow answer:
  - Enumerate all sequences of length t which end in s
  - Add up their probabilities

\[
P(X_t = s) = \sum_{x_1, \ldots, x_{t-1}} P(x_1, \ldots, x_{t-1}, s|X)
\]

Mini-Forward Algorithm

- Better way: cached incremental belief updates

\[
P(x_t) = \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1})
\]

Example

- From initial observation of sun
  \[
  \begin{pmatrix}
  1.0 \\
  0.0
  \end{pmatrix} \rightarrow \begin{pmatrix}
  0.9 \\
  0.1
  \end{pmatrix} \rightarrow \begin{pmatrix}
  0.82 \\
  0.18
  \end{pmatrix} \rightarrow \begin{pmatrix}
  0.5 \\
  0.5
  \end{pmatrix}
  \]
  \[
P(X_1) \quad P(X_2) \quad P(X_3) \quad P(X_4)
  \]

- From initial observation of rain
  \[
  \begin{pmatrix}
  1.0 \\
  0.0
  \end{pmatrix} \rightarrow \begin{pmatrix}
  0.0 \\
  1.0
  \end{pmatrix} \rightarrow \begin{pmatrix}
  0.18 \\
  0.82
  \end{pmatrix} \rightarrow \begin{pmatrix}
  0.5 \\
  0.5
  \end{pmatrix}
  \]
  \[
P(X_1) \quad P(X_2) \quad P(X_3) \quad P(X_4)
  \]
Stationary Distributions

- If we simulate the chain long enough:
  - What happens?
  - Uncertainty accumulates
  - Eventually, we have no idea what the state is!

- Stationary distributions:
  - For most chains, the distribution we end up in is independent of the initial distribution
  - Called the stationary distribution of the chain
  - Usually, can only predict a short time out

Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page (dotted lines)
    - With prob. 1-c, follow a random outlink (solid lines)

- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page!
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors

Most Likely Explanation

- Question: most likely sequence ending in x at t?
  - E.g. if sun on day 4, what’s the most likely sequence?
  - Intuitively: probably sun all four days

- Slow answer: enumerate and score

\[
P(X_t = \text{sun}) = \max_{x_{t-1}} P(x_1, \ldots, x_t, \text{sun})
\]

\[
P(X_t = \text{sun})P(X_{t-1} = \text{sun})P(X_{t-2} = \text{sun})P(X_{t-3} = \text{sun})P(X_3 = \text{sun})P(X_2 = \text{sun})P(X_1 = \text{sun})
\]

... 

Mini-Viterbi Algorithm

- Better answer: cached incremental updates

\[
m_t[x] = \max_{x_{t-1}} P(x_1, \ldots, x_t, x)
\]

\[
s_t[x] = \arg \max_{x_{t-1}} P(x_1, \ldots, x_t, x)
\]

- Read best sequence off of m and a vectors

Mini-Viterbi

\[
m_t[x] = \max_{x_{t-1}} P(x_1, \ldots, x_t, x)
\]

\[
= \max_{x_{t-1}} P(x_1, \ldots, x_{t-1})P(x_t | x_{t-1})
\]

\[
= \max_{x_{t-1}} P(x_t | x_{t-1}) \max_{x_{t-1}} P(x_1, \ldots, x_{t-1})
\]

\[
= \max_{x_{t-1}} P(x_t | x_{t-1}) m_{t-1}[x]
\]

\[
m_1[x] = P(x_1)
\]

Example
Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don’t know anything anymore
  - Need observations to update your beliefs

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states \( S \)
  - You observe outputs (effects) at each time step
  - As a Bayes’ net:

  ![Bayes' Net Diagram]

Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process, future depends on past via the present
  - Current observation independent of all else given current state

  ![Independence Diagram]

  ![Quiz: does this mean that observations are independent given no evidence?]

  - [No, correlated by the hidden state]
Forward Algorithm

- Can ask the same questions for HMMs as Markov chains
- Given current belief state, how to update with evidence?
  - This is called monitoring or filtering
- Formally, we want: \( P(x_t = x | e_{1:t}) \)
  \[
P(x_t | e_{1:t}) \propto P(x_t, e_{1:t}) = \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) = \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1})P(x_t | x_{t-1})P(e_t | x_t) = P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1})P(x_{t-1}, e_{1:t-1})
\]

Viterbi Algorithm

- Question: what is the most likely state sequence given the observations?
  - Slow answer: enumerate all possibilities
  - Better answer: cached incremental version
  \[
r_1^T = \arg \max_{x_1^T} P(x_1:T | e_{1:T}), \quad m_t[x_t] = \max_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) = \max_{x_{t-1}} P(x_{t-1}, e_{1:t-1})P(x_t | x_{t-1})P(e_t | x_t) = P(e_t | x_t) \max_{x_{t-1}} P(x_t | x_{t-1})m_{t-1}[x_{t-1}]
\]

Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation positions (dozens)
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)