Announcements

- Midterm solutions up, regrade requests by 11/13
- Midterm evaluation up, please fill out!
- P5 up, due 11/19
- No section next week
Recap: Some Simple Cases

Models

\[ X_1 \rightarrow E_1 \]

Queries

\[ P(X_1 | e_1) \]

\[ P(X_2 | x_1) \]

\[ P(X_2) \]

\[ P(x_1 | e_1) = \frac{P(x_1, e_1)}{P(e_1)} \]

\[ \propto_{X_1} P(x_1, e_1) \]

\[ = P(x_1)P(e_1 | x_1) \]

\[ P(x_2) = \sum_{x_1} P(x_1, x_2) \]

\[ = \sum_{x_1} P(x_1)P(x_2 | x_1) \]
Hidden Markov Models

- An HMM is
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X|X_{-1})$
  - Emissions: $P(E|X)$
Battleship HMM

- \( P(X_1) = \text{uniform} \)
- \( P(X|X') = \text{usually move according to fixed, known patrol policy (e.g. clockwise), sometimes move in a random direction or stay in place} \)
- \( P(R_{ij}|X) = \text{as before: depends on distance from ships in x to (i,j) (really this is just one of many independent evidence variables that might be sensed)} \)

\[ \begin{array}{ccc} 
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
\end{array} \]

\[ \begin{array}{ccc} 
1/6 & 1/6 & 1/2 \\
0 & 1/6 & 0 \\
0 & 0 & 0 \\
\end{array} \]
Passage of Time

- Assume we have current belief \( P(X | \text{evidence to date}) \)

\[
B(X_t) = P(X_t | e_{1:t})
\]

- Then, after one time step passes:

\[
P(X_{t+1} | e_{1:t}) = \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t})
\]

- Or, compactly:

\[
B'(X_{t+1}) = \sum_{x_t} P(X' | x) B(x_t)
\]

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step the belief is about, and what evidence it includes
Example: Passage of Time

- As time passes, uncertainty “accumulates”

\[ B'(X) = \sum_{x} P(X'|x)B(x) \]

Transition model: ships usually go clockwise.
Assume we have current belief $P(X | \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

Then:

$$P(X_{t+1}|e_{1:t+1}) \propto P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

Or:

$$B(X_{t+1}) \propto P(e|X)B'(X_{t+1})$$

Basic idea: beliefs reweighted by likelihood of evidence

Unlike passage of time, we have to renormalize
Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

\[
B(X) \propto P(e|X)B'(X)
\]

<table>
<thead>
<tr>
<th>Before observation</th>
<th>After observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05 0.01 0.05 &lt;0.01 &lt;0.01 &lt;0.01</td>
<td>&lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01</td>
</tr>
<tr>
<td>0.02 0.14 0.11 0.35 &lt;0.01 &lt;0.01</td>
<td>&lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01</td>
</tr>
<tr>
<td>0.07 0.03 0.05 &lt;0.01 0.03 &lt;0.01</td>
<td>&lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01</td>
</tr>
<tr>
<td>0.03 0.03 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01</td>
<td>&lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01 &lt;0.01</td>
</tr>
</tbody>
</table>
Example HMM
Example HMM
The Forward Algorithm

- We are given evidence at each time and want to know
  \[ B_t(X) = P(X_t|e_{1:t}) \]

- We can derive the following updates
  \[
P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t})
  = \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})
  = \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1})P(x_t|x_{t-1})P(e_t|x_t)
  = P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1}, e_{1:t-1})
\]

We can normalize as we go if we want to have \( P(x|e) \) at each time step, or just once at the end…
Belief Updates

- Every time step, we start with current $P(X | \text{evidence})$
- We update for time:
  
  $$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1}|e_{1:t-1})$$

- We update for evidence:

  $$P(x_t|e_{1:t}) \propto_P e_t|x_t)P(x_t|e_{1:t-1})$$

- The forward algorithm does both at once (and doesn’t normalize)
- Problem: space is $|X|$ and time is $|X|^2$ per time step
Particle Filtering

- Sometimes $|X|$ is too big to use exact inference
  - $|X|$ may be too big to even store $B(X)$
  - E.g. $X$ is continuous
  - $|X|^2$ may be too big to do updates

- Solution: approximate inference
  - Track samples of $X$, not all values
  - Time per step is linear in the number of samples
  - But: number needed may be large

- This is how robot localization works in practice
Particle Filtering: Time

- Each particle is moved by sampling its next position from the transition model

\[ x' = \text{sample}(P(X' | x)) \]

- This is like prior sampling – samples’ frequencies reflect the transition probs
- Here, most samples move clockwise, but some move in another direction or stay in place

- This captures the passage of time
  - If we have enough samples, close to the exact values before and after (consistent)
Particle Filtering: Observation

- Slightly trickier:
  - We don’t sample the observation, we fix it
  - This is similar to likelihood weighting, so we downweight our samples based on the evidence

\[
w(x) = P(e|x)
\]

\[
B(X) \propto P(e|X)B'(X)
\]

- Note that, as before, the probabilities don’t sum to one, since most have been downweighted (in fact they sum to an approximation of \( P(e) \))
Particle Filtering: Resampling

- Rather than tracking weighted samples, we resample.
- N times, we choose from our weighted sample distribution (i.e. draw with replacement).
- This is equivalent to renormalizing the distribution.
- Now the update is complete for this time step, continue with the next one.
Robot Localization

- **In robot localization:**
  - We know the map, but not the robot’s position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store $B(X)$
  - Particle filtering is a main technique

- [DEMOS]
SLAM

- **SLAM = Simultaneous Localization And Mapping**
  - We do not know the map or our location
  - Our belief state is over maps and positions!
  - Main techniques: Kalman filtering (Gaussian HMMs) and particle methods

- [DEMOS]

DP-SLAM, Ron Parr