1 Jabberwock in the wild

Lewis’ Jabberwock is in the wild: its position is in a two-dimensional discrete grid, but this time the grid is not bounded. In other words, the position of the Jabberwock is a pair of integers \(z = (x, y) \in \mathbb{Z}^2 = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \times \{\ldots, -2, -1, 0, 1, 2, \ldots\}\). At each time step \(t = 1, 2, 3, \ldots\), the Jabberwock is in some cell \(Z_t = z \in \mathbb{Z}^2\), and it moves to cell \(Z_{t+1}\) randomly as follows: with probability 1/2, it stays where it is, otherwise, it chooses one of the four neighboring cells uniformly at random (fortunately, no teleportation is allowed this week!).

(a) Write the transition probabilities.

At each time step \(t\) you get an observation of the x coordinate \(R_t\) in which the Jabberwock sits, but it is a noisy observation. Given that the true position is \(Z_t = (x, y)\), you observe the correct value \(R_t = x\) half of the time, but the other half of the time, the observed value \(R_t = r\) is different from \(x\), with the following distribution: the absolute value of the difference between \(x\) and \(r\) is geometrically distributed with mean 2, and the difference has equal chance of being positive or negative. This means:

\[
P(R_t = r|Z_t = (x, y)) = \begin{cases} \frac{1}{2} & \text{if } r = x \\ \frac{1}{4} \left(\frac{1}{2}\right)^{|x-r|-1} & \text{otherwise} \end{cases}
\]

To track the Jabberwock, we will use a particle filter.

In each question, use as many values as needed from the following sequence \(a_i\) of numbers generated independently and uniformly at random from \([0, 1]\) as a source of randomness. Restart at \(a_1\) in each subproblem so that you don’t run out.

<table>
<thead>
<tr>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
<th>(a_5)</th>
<th>(a_6)</th>
<th>(a_7)</th>
<th>(a_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.142</td>
<td>0.522</td>
<td>0.916</td>
<td>0.792</td>
<td>0.036</td>
<td>0.859</td>
<td>0.656</td>
<td>0.149</td>
</tr>
</tbody>
</table>

You can use these values to sample from any discrete distribution of the form \(P(X)\) where \(X\) takes values in \(\{1, 2, \ldots, N\}\). Given \(a \sim U[0, 1]\), return \(j\) such that \(\sum_{k=1}^{j-1} P(X = k) \leq a < \sum_{k=1}^j P(X = k)\). In other words, return the index where the CDF becomes larger than the random \(a\) that you drew. (You have to fix an ordering of the elements for this procedure to make sense.)

(b) We now review how to do particle resampling. Suppose that at some time step \(t\) there are 4 particles, \(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}\) in our particle filter, with weights 0.25, 0.25, 1.0, 0.5 say after observation reweighting. Use them to get 4 unweighted particles. Access the particle weights in the order that they appear (other orders are also correct, but this way makes it easier to compare your answer with the solution).
Suppose that you know that half of the time, the Jabberwock starts at \( z_1 = (0,0) \), and the other half, at \( z_1 = (1,1) \). Now, you get the following observations: \( R_1 = 1, R_2 = 0, R_3 = 1 \).

(c) Use a particle filter with 2 particles to compute an approximation to \( P(Z_1, Z_2, Z_3 | R_1 = 1, R_2 = 0, R_3 = 1) \). Note that we are asking for the joint distribution of these three variables, not the marginal of \( Z_3 \). For transitions, use the provided independent random samples distributed as follows: “stay” with probability 1/2, and “up”, “down”, “left”, “right” with probability 1/8 each.

<table>
<thead>
<tr>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( t_4 )</th>
<th>( t_5 )</th>
<th>( t_6 )</th>
<th>( t_7 )</th>
<th>( t_8 )</th>
<th>( t_9 )</th>
<th>( t_{10} )</th>
<th>( t_{11} )</th>
<th>( t_{12} )</th>
<th>( t_{13} )</th>
<th>( t_{14} )</th>
<th>( t_{15} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>left</td>
<td>stay</td>
<td>left</td>
<td>stay</td>
<td>right</td>
<td>stay</td>
<td>right</td>
<td>left</td>
<td>stay</td>
<td>down</td>
<td>stay</td>
<td>stay</td>
<td>down</td>
<td>down</td>
<td></td>
</tr>
</tbody>
</table>

Hint: Use \( a_1, a_2 \) to sample from the prior; to check that you are in the right track, at the end of step \( t = 1 \), you should have two particles at \((1,1)\); at the end of step \( t = 2 \), the first particle should be at \((0,1)\) and the second at \((1,1)\).
(d) Use your samples (the unweighted particles in the last step) to evaluate the posterior probability that the x-coordinate of $Z_3$ is different than the column of $Z_3$, i.e. $X_3 \neq Y_3$.

(e) Use your samples to evaluate the expected number of distinct x-coordinates visited by the Jabberwock in these 3 time steps.

(f) Use your samples to evaluate the posterior probability that the Jabberwock returned to its starting point.

(g) Instead of using the unweighted particles, can you use the weighted particles of the last step to get an estimate of these quantities? How?

(h) Could you use the elimination algorithm to compute these quantities?