Maximum Expected Utility

- Why should we average utilities? Why not minimax?
- Principle of maximum expected utility:
  - A rational agent should choose the action which maximizes its expected utility, given its knowledge
- Questions:
  - Where do utilities come from?
  - How do we know such utilities even exist?
  - Why are we taking expectations of utilities (not, e.g. minimax)?
  - What if our behavior can’t be described by utilities?

Utilities

- Utilities are functions from outcomes (states of the world) to real numbers that describe an agent’s preferences
- Where do utilities come from?
  - In a game, may be simple (+1/-1)
  - Utilities summarize the agent’s goals
- Theorem: any rational preferences can be summarized as a utility function
- We hard-wire utilities and let behaviors emerge
  - Why don’t we let agents pick utilities?
  - Why don’t we prescribe behaviors?

Utilities: Uncertain Outcomes

Preferences

- An agent must have preferences among:
  - Prizes: A, B, etc.
  - Lotteries: situations with uncertain prizes
    \[ L = [p, A; (1-p), B] \]
- Notation:
  - \( A \succ B \) \( A \) preferred over \( B \)
  - \( A \sim B \) indifference between \( A \) and \( B \)
  - \( A \succeq B \) \( B \) not preferred over \( A \)

Rational Preferences

- We want some constraints on preferences before we call them rational
  \( (A \succ B) \land (B \succ C) \Rightarrow (A \succ C) \)
- For example: an agent with intransitive preferences can be induced to give away all of its money
  - If \( B \succ C \), then an agent with \( C \)
    would pay (say) 1 cent to get \( B \)
  - If \( A \succ B \), then an agent with \( B \)
    would pay (say) 1 cent to get \( A \)
  - If \( C \succ A \), then an agent with \( A \)
    would pay (say) 1 cent to get \( C \)
Rational Preferences

Preferences of a rational agent must obey constraints.
- The axioms of rationality:
  - Orderability: \((A > B) \lor (B > A) \lor (A \sim B)\)
  - Transitivity: \((A > B) \land (B > C) \Rightarrow (A > C)\)
  - Continuity: \(A \sim B \sim C \Rightarrow \exists \alpha \in [0, 1] : A \sim \alpha B \sim (1-\alpha) C\)
  - Substitutability: \(A > B \Rightarrow \exists \alpha \in [0, 1] : A \sim \alpha B \sim (1-\alpha) C\)
  - Monotonicity: \(A > B \Rightarrow \exists \alpha \in [0, 1] : A \sim \alpha B \sim (1-\alpha) C\)
- Theorem: Rational preferences imply behavior describable as maximization of expected utility

MEU Principle

- Theorem:
  - [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Given any preferences satisfying these constraints, there exists a real-valued function \(U\) such that:
    \[ U(A) > U(B) \iff A \succ B \]
    \[ U\left(\sum_{i=1}^{n} p_i S_i\right) = \sum_{i=1}^{n} p_i U(S_i) \]
- Maximum expected utility (MEU) principle:
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tic-tac-toe, reflex vacuum cleaner

Utility Scales

- Normalized utilities: \(u_+ = 1.0, u_- = 0.0\)
- Micromorts: one-millionth chance of death, useful for paying to reduce product risks, etc.
- QALYs: quality-adjusted life years, useful for medical decisions involving substantial risk
- Note: behavior is invariant under positive linear transformation
  \[ U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0 \]
- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes

Human Utilities

- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment of human utilities:
  - Compare a state \(A\) to a standard lottery \(L_p\) between
    - \(l_{\text{best possible prize}}\) with probability \(p\)
    - \(l_{\text{worst possible catastrophe}}\) with probability \(1-p\)
  - Adjust lottery probability \(p\) until \(A \sim L_p\)
  - Resulting \(p\) is a utility in \([0,1]\)

Example: Insurance

- Consider the lottery \([0.5,$1000; 0.5,$0]\)
- What is its expected monetary value? ($500)
- What is its certainty equivalent?
- Monetary value acceptable in lieu of lottery
- $400 for most people
- Difference of $100 is the insurance premium
- There’s an insurance industry because people will pay to reduce their risk
- If everyone were risk-neutral, no insurance needed!
Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8,$4k; 0.2,$0]
  - B: [1.0,$3k; 0.0,$0]
  - C: [0.2,$4k; 0.8,$0]
  - D: [0.25,$3k; 0.75,$0]

- Most people prefer B > A, C > D
- But if $U(0) = 0$, then
  - B > A \Rightarrow U($3k) > 0.8 U($4k)
  - C > D \Rightarrow 0.8 U($4k) > U($3k)