Q2. Variable Elimination

(a) For the Bayes’ net below, we are given the query $P(A, E \mid +c)$. All variables have binary domains. Assume we run variable elimination to compute the answer to this query, with the following variable elimination ordering $B, D, G, F$.

![Bayes' net diagram]

Complete the following description of the factors generated in this process:
After inserting evidence, we have the following factors to start out with:

\[ P(A), P(B \mid A), P(\neg c), P(D \mid A, B, +c), P(E \mid D), P(F \mid D), P(G \mid +c, F) \]

When eliminating $B$ we generate a new factor $f_1$ as follows:

\[ f_1(A, +c, D) = \sum_B P(B \mid A)P(D \mid A, b, +c) \]

This leaves us with the factors:

\[ P(A), P(+c), P(E \mid D), P(F \mid D), P(G \mid +c, F), f_1(A, +c, D) \]

When eliminating $D$ we generate a new factor $f_2$ as follows:

This leaves us with the factors:

When eliminating $G$ we generate a new factor $f_3$ as follows:

This leaves us with the factors:
\[ P(A = +a) \text{ is straightforward} \]

- We already know \( P(A) \)

\[ P(F | C) \text{ not so straightforward} \]

- So we'll try to express it in terms of things we do know.
- Since we already have conditional probabilities, we'll use Bayes' rule.

\[ P(F | C) = \frac{P(F, C)}{P(C)} \]

- We already knew \( P(C) \) so that leaves us with \( P(F, C) \).
- How far away from \( C \). What do we do?
- Eliminate variables.
- Variable elimination

\[ P(F, C) = \frac{P(F, C)}{P(C)} \rightarrow \sum_{C} \]

\[ \star: P(F, C) = \sum_{A} P(A, B, C, D, E, F, G) \]

- Let's introduce new variables and then marginalize them.

\[ \star: P(F, C) = \sum_{A} P(A) P(B | A) P(C \mid D) P(D | E) P(E | F) P(F | G, C, F) \]

- Select ordering that minimizes maximum factor size
- What
- Now we'll eliminate variables
- Order in which we eliminate matters.
- First let's eliminate \( G \).

\[ \star: P(F, C) = \sum_{A} P(A) P(B | A) P(C \mid D) P(D | E) P(E | F) P(F | G, C, F) \]

\[ = \sum_{A} P(A) P(B | A) P(C \mid D) P(D | E) P(E | F) \sum_{G} P(G | C, F) \]

- Only term that relies on \( G \) is the last one: \( P(G | C, F) \)
- Because the only unconditioned variable in \( P(G | C, F) \),
  the sum \( \sum_{G} P(G | C, F) = 1 \)

\[ \star: P(F, C) = \sum_{A} P(A) P(B | A) P(C \mid D) P(D | E) P(E | F) \]

- Now let's eliminate \( E \).

\[ \star: P(F, C) = \sum_{A} P(A) P(B | A) P(C \mid D) P(D | E) P(E | F) \sum_{E} P(E | F) \]

- Same rule applies, \( \sum_{E} P(E | F) = 1 \) but only second var is \( E \).
\[ P(F|C) = \sum_{AB} P(A) P(B|A) P(C) P(D|A,B,C) P(F|D) \]

- Now let's eliminate \( F \).
  - WAIT NO! We can't blc \( F \) is part of what we're trying to find.
  - Same with \( C \).
  - So instead let's eliminate \( D \). (We can choose an easier factor but we want to make our lives hard)

\[ P(F|C) = \sum_{AB} P(A) P(B|A) P(C) \sum_{D} P(D|A,B,C) P(F|D) \]

- Now we're not so lucky.
  1. There are 2 terms we're summing over.
  2. \( D \) appears as a conditioned variable.
- So we'll turn \( D \) into a factor. (Factors are pretty much functions)

\[
\sum_{D} P(D|A,B,C) P(F|D) = F_1(A,B,C,F)
\]

Can't be 1 blc weighted by \( P(F|D) \)

\[ P(F|C) = \sum_{AB} P(A) P(B|A) P(C) F_1(A,B,C,F) \]

- Eliminate \( B \) next.

\[ P(F,C) = \sum_{A} P(A) P(C) \sum_{B} P(B|A) F_1(A,B,C,F) \]

- Make another factor.
  - Remember factors don't depend on variables you're summing over.

\[
\sum_{B} P(B|A) F_1(A,B,C,F) = F_2(A,C,F)
\]

\[ P(F,C) = \sum_{A} P(A) P(C) F_2(A,C,F) \]

- Eliminate \( A \).

\[ P(F,C) = P(C) \sum_{A} P(A) F_2(A,C,F) \]

- Make another factor.

\[
\sum_{A} P(A) F_2(A,C,F) = F_3(C,F)
\]

\[ P(F,C) = P(C) F_3(C,F) \]

- We did it.
Let’s evaluate $P(F = -f, C = +c)$

$$= P(C = +c) F_3(C = +c, F = -f)$$

$$= P(C = +c) \sum_A P(A) F_2(A, C = +c, F = -f)$$

$$= P(C = +c) \sum_A P(A) \sum_B P(B|A) F_1(A, B, C = +c, F = -f)$$

$$= P(C = +c) \sum_A P(A) \sum_B P(B|A) \sum_D P(D|A, B, C = +c) P(F = -f|D)$$

- Factor size: $F_1(A, B, C, F)$
  - 4 variables
  - $2^4$ table size

- Select ordering that minimizes maximum factor size
- wait

- How can we do this?
  - Sum out variables that sum to one.
  - Like $P(G|C, F)$ and $P(G|D)$.

<table>
<thead>
<tr>
<th>Elimination Order</th>
<th>Max Factor Size</th>
</tr>
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<tbody>
<tr>
<td>GEDBA</td>
<td>4</td>
</tr>
<tr>
<td>DBAEG</td>
<td>$?_1$</td>
</tr>
<tr>
<td>BAEEDG</td>
<td>$?_2$</td>
</tr>
<tr>
<td>...</td>
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Minimize this