Trees naturally represent recursively defined, hierarchical objects with more than one recursive subpart for each instance.

- Common examples: expressions, sentences.
- Expressions have definitions such as "an expression consists of a literal or two expressions separated by an operator."
- Also describe structures in which we recursively divide a set into multiple subsets.

Fundamental Operation: Traversal

- **Traversing a tree** means enumerating (some subset of) its nodes.
- Typically done recursively, because that is natural description.
- As nodes are enumerated, we say they are visited.
- Three basic orders for enumeration (+ variations):
  - **Preorder:** visit node, traverse its children.
  - **Postorder:** traverse children, visit node.
  - **Inorder:** traverse first child, visit node, traverse second child (binary trees only).

Preorder Traversal and Prefix Expressions

Problem: Convert

```
- (- (* x (+ y 3))) z
```

(Tree<Label> is supposed to mean "Tree whose labels have type Label.")

```
static String toLisp (Tree<String> T) {
    if (T == null)
        return "";
    else if (T.degree () == 0)
        return T.label ();
    else {
        String R; R = "";
        for (int i = 0; i < T.numChildren (); i += 1)
            R += " " + toLisp (T.child (i));
        return String.format ("(%s)", T.label (), R);
    }
```
Inorder Traversal and Infix Expressions

Problem: Convert

\[-(x(y+3))-z\]

To think about: how to get rid of all those parentheses.

```java
static String toInfix (Tree<String> T) {
    if (T == null)
        return "";
    if (T.degree () == 0)
        return T.label ();
    else {
        return String.format ("(%s%s%s)",
            toInfix (T.left ()), T.label (), toInfix (T.right ())
    }
}
```

Postorder Traversal and Postfix Expressions

Problem: Convert

\[x \ y \ 3 \ +:2 \ *:2 \ -:1 \ z \ -:2\]

```java
static String toPolish (Tree<String> T) {
    if (T == null)
        return "";
    String R; R = "";
    for (int i = 0; i < T.numChildren (); i += 1)
        R += toPolish (T.child (i)) + " ";
    return String.format ("%s%sa", R, T.label (), T.degree ());
}
```

A General Traversal: The Visitor Pattern

```java
void preorderTraverse (Tree<Label> T, Action<Label> whatToDo)
{
    if (T != null) {
        whatToDo.action (T);
        for (int i = 0; i < T.numChildren (); i += 1)
            preorderTraverse (T.child (i), whatToDo);
    }
}
```

- What is Action?

```java
interface Action<Label> {
    void action (Tree<Label> T);
}
```

```java
class Print implements Action<String> {
    void action (Tree<String> T) {
        System.out.print (T.label ());
    }
}
```

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The traversal algorithms have roughly the form of the boom example in §1.3.3 of Data Structures—an exponential algorithm.

However, the role of $M$ in that algorithm is played by the height of the tree, not the number of nodes.

In fact, easy to see that tree traversal is linear: $\Theta(N)$, where $N$ is the # of nodes: Form of the algorithm implies that there is one visit at the root, and then one visit for every edge in the tree. Since every node but the root has exactly one parent, and the root has none, must be $N-1$ edges in any non-empty tree.

In positional tree, is also one recursive call for each empty tree, but # of empty trees can be no greater than $kN$, where $k$ is arity.

For $k$-ary tree (max # children is $k$), $h + 1 \leq N \leq \frac{k^{h+1}-1}{k-1}$, where $h$ is height.

So $h \in \Omega(\log_k N) = \Omega(\log N)$ and $h \in O(N)$.

Many tree algorithms look at one child only. For them, time is proportional to the height of the tree, and this is $\Theta(\log N)$, assuming that tree is bushy—each level has about as many nodes as possible.

Iterative Deepening Time?

Let $h$ be height, $N$ be # of nodes.

Count # edges traversed (i.e., # of calls, not counting null nodes).

First (full) tree: 1 for level 0, 3 for level 1, 7 for level 2, 15 for level 3.

Or in general $(2^1 - 1) + (2^2 - 1) + \ldots + (2^{h+1} - 1) = 2^{h+2} - h \in \Theta(N)$, since $N = 2^{h+1} - 1$ for this tree.

Second (right leaning) tree: 1 for level 0, 2 for level 2, 3 for level 3.

Or in general $(h+1)(h+2)/2 = N(N+1)/2 \in \Theta(N^2)$, since $N = h+1$ for this kind of tree.

Iterative Traversals

Tree recursion conceals data: a stack of nodes (all the T arguments) and a little extra information. Can make the data explicit, e.g:

```
void preorderTraverse2 (Tree<Label> T, Action whatToDo) {
    Stack<Label> s = new Stack<Label> ();
    s.push (T);
    while (! s.isEmpty ()) {
        Tree<Label> node = s.pop ();
        if (node == null)
            continue;
        whatToDo.action (node);
        for (int i = node.numChildren (); i >= 0; i -= 1)
            s.push (node.child (i));
    }
}
```

To do a breadth-first traversal, use a queue instead of a stack, replace push with add, and pop with removeFirst.

Makes breadth-first traversal worst-case linear time in all cases, but also linear space for "bushy" trees.
Iterators for Trees

• Frankly, iterators are not terribly convenient on trees.
• But can use ideas from iterative methods.

```java
class PreorderTreeIterator<Label> implements Iterator<Label> {
    private Stack<Tree<Label>> s = new Stack<Tree<Label>>();

    public PreorderTreeIterator (Tree<Label> T) { s.push (T); }

    public boolean hasNext () { return ! s.isEmpty (); }
    public T next () {
        Tree<Label> result = s.pop ();
        for (int i = result.numChildren ()-1; i >= 0; i -= 1)
            s.push (result.child (i));
        return result.label ();
    }
    void remove () { throw new UnsupportedOperationException (); }
}
```

Example: (what do I have to add to class Tree first?)

```java
for (String label : aTree) System.out.print (label + " ");
```

Tree Representation

(a) Embedded child pointers (+ optional parent pointers)
(b) Array of child pointers (+ optional parent pointers)
(c) Child/sibling pointers
(d) Breadth-first array (complete trees)

Divide and Conquer

• Much (most?) computation is devoted to finding things in response to various forms of query.
• Linear search for response can be expensive, especially when data set is too large for primary memory.
• Preferable to have criteria for dividing data to be searched into pieces recursively
• Remember figure for $\log N$ algorithms: at 1$\mu$s/comp, could process $10^{300000}$ items in 1 sec.
• Tree is a natural framework for the representation:

```
decision data
/ \
no yes
```

Binary Search Trees

Binary Search Property:

• Tree nodes contain keys, and possibly other data.
• All nodes in left subtree of node have smaller keys.
• All nodes in right subtree of node have larger keys.
• "Smaller" means any complete transitive, anti-symmetric ordering on keys:
  - exactly one of $x < y$ and $y < x$ true.
  - $x < y$ and $y < z$ imply $x < z$.
  - (To simplify, won't allow duplicate keys this semester).
• E.g., in dictionary database, node label would be (word, definition): word is the key.
Finding

- Searching for 50 and 49:

```java
/** Node in T containing L, * or null if none */
static BST find(BST T, Key L) {
    if (T == null)
        return T;
    if (L.keyequals (T.label()))
        return T;
    else if (L ≺ T.label())
        return find(T.left(), L);
    else
        return find(T.right (), L);
}
```

- Dashed boxes show which node labels we look at.
- Number looked at proportional to height of tree.

Inserting

- Inserting 27

```java
/** Insert L in T, replacing existing * value if present, and returning * new tree. */
BST insert(BST T, Key L) {
    if (T == null)
        return new BST(L);
    if (L.keyequals (T.label()))
        T.setLabel (L);
    else if (L ≺ T.label())
        T.setLeft(insert (T.left (), L));
    else
        T.setRight(insert (T.right (), L));
    return T;
}
```

- Starred edges are set (to themselves, unless initially null).
- Again, time proportional to height.

Deletion

```
Initial

Remove 27

formerly contained 42
```

A Leap Ahead: Quadtrees

- Want to index information about locations so that items can be retrieved by position.
- Quadtrees do so using standard data-structuring trick: Divide and Conquer.
- Idea: divide (2D) space into four quadrants, and store items in the appropriate quadrant. Repeat this recursively with each quadrant that contains more than one item.
- Original definition: a quadtree is either
  - Empty, or
  - An item at some position \((x, y)\), called the root, plus
  - four quadtrees, each containing only items that are northwest, northeast, southwest, and southeast of \((x, y)\).
- Big idea is that if you are looking for point \((x', y')\) and the root is not the point you are looking for, you can narrow down which of the four subtrees of the root to look in by comparing coordinates \((x, y)\) with \((x', y')\).
Classical Quadtree: Example

Point-region (PR) Quadtrees

- If we use a QuadTree to track moving objects, it may be useful to be able to delete items from a tree: when an object moves, the subtree that it goes in may change.
- Difficult to do with the classical data structure above, so we’ll define instead:
  - A quadtree consists of a bounding rectangle, \( B \) and either
    - Zero up to a small number of items that lie in that rectangle, or
    - Four quadtrees whose bounding rectangles are the four quadrants of \( B \) (all of equal size).
  - A completely empty quadtree can have an arbitrary bounding rectangle, or you can wait for the first point to be inserted.

Example of PR Quadtree

Navigating PR Quadtrees

- To find an item at \((x, y)\) in quadtree \(T\),
  1. If \((x, y)\) is outside the bounding rectangle of \(T\), or \(T\) is empty, then \((x, y)\) is not in \(T\).
  2. Otherwise, if \(T\) contains a small set of items, then \((x, y)\) is in \(T\) iff it is among these items.
  3. Otherwise, \(T\) consists of four quadtrees. Recursively look for \((x, y)\) in each (however, step #1 above will cause all but one of these bounding boxes to reject the point immediately).
- Similar procedure works when looking for all items within some rectangle, \(R\):
  1. If \(R\) does not intersect the bounding rectangle of \(T\), or \(T\) is empty, then there are no items in \(R\).
  2. Otherwise, if \(T\) contains a set of items, return those that are in \(R\), if any.
  3. Otherwise, \(T\) consists of four quadtrees. Recursively look for points in \(R\) in each one of them.
Insertion into PR Quadtrees

Various cases for inserting a new point $N$, showing initial state $\implies$ final state.

- Initial state:
  - Top left: $N$ at $(0,0)$
  - Top right: $N$ at $(10,10)$

- Final state:
  - Top left: $N$ at $(0,0)$
  - Top right: $N$ at $(10,10)$

- Additional cases:
  - Bottom left:
    - Initial state: $N$ at $(0,0)$
    - Final state: $N$ at $(0,0)$
  - Bottom right:
    - Initial state: $N$ at $(5,5)$
    - Final state: $N$ at $(5,5)$