Today:

- Sorting algorithms: why?
- Insertion, Shell’s, Heap, Merge sorts
- Quicksort
- Selection
- Distribution counting, radix sorts

Readings: Today: DS(IJ), Chapter 8; Next topic: Chapter 9.
Purposes of Sorting

- Sorting supports searching
- Binary search standard example
- Also supports other kinds of search:
  - Are there two equal items in this set?
  - Are there two items in this set that both have the same value for property X?
  - What are my nearest neighbors?
- Used in numerous unexpected algorithms, such as convex hull (smallest convex polygon enclosing set of points).
Some Definitions

• A sort is a permutation (re-arrangement) of a sequence of elements that brings them into order, according to some total order. A total order, $\preceq$, is:
  
  - Total: $x \preceq y$ or $y \preceq x$ for all $x, y$.
  - Reflexive: $x \preceq x$.
  - Antisymmetric: $x \preceq y$ and $y \preceq x$ iff $x = y$.
  - Transitive: $x \preceq y$ and $y \preceq z$ implies $x \preceq z$.

• However, our orderings may allow unequal items to be equivalent:
  
  - E.g., can be two dictionary definitions for the same word: if entries sorted only by word, then sorting could put either entry first.
  - A sort that does not change the relative order of equivalent entries is called stable.
Classifications

- **Internal sorts** keep all data in primary memory
- **External sorts** process large amounts of data in batches, keeping what won’t fit in secondary storage (in the old days, tapes).
- **Comparison-based sorting** assumes only thing we know about keys is order
- **Radix sorting** uses more information about key structure.
- **Insertion sorting** works by repeatedly inserting items at their appropriate positions in the sorted sequence being constructed.
- **Selection sorting** works by repeatedly selecting the next larger (smaller) item in order and adding it one end of the sorted sequence being constructed.
Sorting by Insertion

- Simple idea:
  - starting with empty sequence of outputs.
  - add each item from input, inserting into output sequence at right point.

- Very simple, good for small sets of data.

- With vector or linked list, time for find + insert of one item is at worst $\Theta(k)$, where $k$ is # of outputs so far.

- So gives us $O(N^2)$ algorithm. Can we say more?
Inversions

• Can run in $\Theta(N)$ comparisons if already sorted.

• Consider a typical implementation for arrays:

```java
for (int i = 1; i < A.length; i += 1) {
    int j;
    Object x = A[i];
    for (j = i-1; j >= 0; j -= 1) {
        if (A[j].compareTo(x) <= 0) /* (1) */
            break;
    }
    A[j+1] = x;
}
```

• #times (1) executes $\approx$ how far $x$ must move.

• If all items within $K$ of proper places, then takes $O(KN)$ operations.

• Thus good for any amount of nearly sorted data.

• One measure of unsortedness: # of inversions: pairs that are out of order (= 0 when sorted, $N(N-1)/2$ when reversed).

• Each step of $j$ decreases inversions by 1.
Shell’s sort

Idea: Improve insertion sort by first sorting distant elements:

• First sort subsequences of elements $2^k - 1$ apart:
  - sort items $\#0, 2^k - 1, 2(2^k - 1), 3(2^k - 1), \ldots$, then
  - sort items $\#1, 1 + 2^k - 1, 1 + 2(2^k - 1), 1 + 3(2^k - 1), \ldots$, then
  - sort items $\#2, 2 + 2^k - 1, 2 + 2(2^k - 1), 2 + 3(2^k - 1), \ldots$, then
  - etc.
  - sort items $\#2^k - 2, 2(2^k - 1) - 1, 3(2^k - 1) - 1, \ldots$,
  - Each time an item moves, can reduce #inversions by as much as $2^k + 1$.

• Now sort subsequences of elements $2^{k-1} - 1$ apart:
  - sort items $\#0, 2^{k-1} - 1, 2(2^{k-1} - 1), 3(2^{k-1} - 1), \ldots$, then
  - sort items $\#1, 1 + 2^{k-1} - 1, 1 + 2(2^{k-1} - 1), 1 + 3(2^{k-1} - 1), \ldots$,
  -:

• End at plain insertion sort ($2^0 = 1$ apart), but with most inversions gone.

• Sort is $\Theta(N^{1.5})$ (take CS170 for why!).
Example of Shell's Sort

$I$: Inversions left.  
$C$: Comparisons needed to sort subsequences.
Sorting by Selection: Heapsort

**Idea:** Keep selecting smallest (or largest) element.

- Really bad idea on a simple list or vector.
- But we've already seen it in action: use heap.
- Gives $O(N \lg N)$ algorithm ($N$ remove-first operations).
- Since we remove items from end of heap, we can use that area to accumulate result:

```
original:  19  0  -1  7  23  2  42
heapified:  42  23  19  7  0  2  -1
  23  7  19  -1  0  2  42
  19  7  2  -1  0  23  42
  7  0  2  -1  19  23  42
  2  0  -1  7  19  23  42
  0  -1  2  7  19  23  42
  -1  0  2  7  19  23  42
```
Merge Sorting

**Idea:** Divide data in 2 equal parts; recursively sort halves; merge results.

- Already seen analysis: $\Theta(N \lg N)$.
- Good for external sorting:
  - First break data into small enough chunks to fit in memory and sort.
  - Then repeatedly merge into bigger and bigger sequences.
  - Can merge $K$ sequences of arbitrary size on secondary storage using $\Theta(K)$ storage.
- For internal sorting, can use binomial comb to orchestrate:
Illustration of Internal Merge Sort

L: (9, 15, 5, 3, 0, 6, 10, -1, 2, 20, 8)

```
0: 0
1: 0
2: 0
3: 0
0 elements processed

0: 1 → (9)
1: 0
2: 0
3: 0
1 element processed

0: 0
1: 1 → (9, 15)
2: 0
3: 0
2 elements processed

0: 0
1: 1 → (9, 15)
2: 0
3: 0
3 elements processed

0: 1 → (3, 5, 9, 15)
1: 0
2: 1
3: 0
4 elements processed

0: 0
1: 1 → (0, 6)
2: 1 → (3, 5, 9, 15)
3: 0
5 elements processed

0: 1 → (5)
1: 1 → (9, 15)
2: 0
3: 0
6 elements processed

0: 1 → (8)
1: 1 → (2, 20)
2: 0
3: 1 → (-1, 0, 3, 5, 6, 9, 10, 15)
11 elements processed
```
QuickSort: Speed through Probability

Idea:

• **Partition** data into pieces: everything \( > \) a pivot value at the high end of the sequence to be sorted, and everything \( \leq \) on the low end.

• Repeat recursively on the high and low pieces.

• For speed, stop when pieces are “small enough” and do insertion sort on the whole thing.

• Reason: insertion sort has low constant factors. By design, no item will move out of its piece [why?], so when pieces are small, \#inversions is, too.

• Have to choose pivot well. E.g.: median of first, last and middle items of sequence.
Example of Quicksort

- In this example, we continue until pieces are size $\leq 4$.
- Pivots for next step are starred. Arrange to move pivot to dividing line each time.
- Last step is insertion sort.

16 10 13 18 -4 -7 12 -5 19 15 0 22 29 34 -1*

-4 -5 -7 -1 18 13 12 10 19 15 0 22 29 34 16*

-4 -5 -7 -1 15 13 12* 10 0 16 19* 22 29 34 18

-4 -5 -7 -1 10 0 12 15 13 16 18 19 29 34 22

- Now everything is “close to” right, so just do insertion sort:

-7 -5 -4 -1 0 10 12 13 15 16 18 19 22 29 34
Performance of Quicksort

• Probabilistic time:
  - If choice of pivots good, divide data in two each time: $\Theta(N \log N)$
    with a good constant factor relative to merge or heap sort.
  - If choice of pivots bad, most items on one side each time: $\Theta(N^2)$.
  - $\Omega(N \log N)$ in best case, so insertion sort better for nearly ordered input sets.

• Interesting point: randomly shuffling the data before sorting makes $\Omega(N^2)$ time very unlikely!
Quick Selection

The Selection Problem: for given $k$, find $k^{th}$ smallest element in data.

- Obvious method: sort, select element $\#k$, time $\Theta(N \log N)$.
- If $k \leq$ some constant, can easily do in $\Theta(N)$ time:
  - Go through array, keep smallest $k$ items.
- Get probably $\Theta(N)$ time for all $k$ by adaptting quicksort:
  - Partition around some pivot, $p$, as in quicksort, arrange that pivot ends up at dividing line.
  - Suppose that in the result, pivot is at index $m$, all elements $\leq$ pivot have indices $\leq m$.
  - If $m = k$, you're done: $p$ is answer.
  - If $m > k$, recursively select $k^{th}$ from left half of sequence.
  - If $m < k$, recursively select $(k - m - 1)^{th}$ from right half of sequence.
Selection Example

Problem: Find just item #10 in the sorted version of array:

Initial contents:

| 51 | 60 | 21 | -4 | 37 | 4 | 49 | 10 | 40* | 59 | 0 | 13 | 2 | 39 | 11 | 46 | 31 |
|----|----|----|----|----|---|---|----|----|----|---|---|---|---|----|----|----|----|

Looking for #10 to left of pivot 40:

| 13 | 31 | 21 | -4 | 37 | 4* | 11 | 10 | 39 | 2 | 0 | 40 | 59 | 51 | 49 | 46 | 60 |
|----|----|----|----|----|----|----|----|----|---|---|----|---|----|----|----|----|----|

Looking for #6 to right of pivot 4:

| -4 | 0 | 2 | 4 | 37 | 13 | 11 | 10 | 39 | 21 | 4* | 0 | 40 | 59 | 51 | 49 | 46 | 60 |
|----|---|---|---|---|----|----|----|----|----|----|---|----|---|----|----|----|----|----|

Looking for #1 to right of pivot 31:

| -4 | 0 | 2 | 4 | 21 | 13 | 11 | 10 | 31 | 39 | 37 | 40 | 59 | 51 | 49 | 46 | 60 |
|----|---|---|---|----|----|----|----|----|----|----|----|---|---|----|----|----|----|----|

Just two elements: just sort and return #1:

| -4 | 0 | 2 | 4 | 21 | 13 | 11 | 10 | 31 | 37 | 39 | 40 | 59 | 51 | 49 | 46 | 60 |
|----|---|---|---|----|----|----|----|----|----|----|----|---|---|----|----|----|----|----|

Result: 39
Selection Performance

• For this algorithm, if $m$ roughly in middle each time, cost is

$$C(N) = \begin{cases} 
1, & \text{if } N = 1, \\
N + C(N/2), & \text{otherwise.}
\end{cases}$$

$$= N + N/2 + \ldots + 1$$

$$= 2N - 1 \in \Theta(N)$$

• But in worst case, get $\Theta(N^2)$, as for quicksort.

• By another, non-obvious algorithm, can get $\Theta(N)$ worst-case time for all $k$ (take CS170).
Better than $N \lg N$?

- Can prove that if all you can do to keys is compare them then sorting must take $\Omega(N \lg N)$.
- Basic idea: there are $N!$ possible ways the input data could be scrambled.
- Therefore, your program must be prepared to do $N!$ different combinations of move operations.
- Therefore, there must be $N!$ possible combinations of outcomes of all the if tests in your program (we're assuming that comparisons are 2-way).
- Since each if test goes two ways, number of possible different outcomes for $k$ if tests is $2^k$.
- Thus, need enough tests so that $2^k > N!$, which means $k \in \Omega(\lg N!)$.
- Using Stirling’s approximation,

$$m! \in \sqrt{2\pi m} \left(\frac{m}{e}\right)^m \left(1 + \Theta\left(\frac{1}{m}\right)\right),$$

this tells us that

$$k \in \Omega(N \lg N).$$
Beyond Comparison: Distribution Counting

• But suppose can do more than compare keys?

• For example, how can we sort a set of $N$ integer keys whose values range from 0 to $kN$, for some small constant $k$?

• One technique: count the number of items $< 1, < 2$, etc.

• If $M_p = \#\text{items with value } < p$, then in sorted order, the $j^{th}$ item with value $p$ must be $\#M_p + j$.

• Gives linear-time algorithm.
Distribution Counting Example

• Suppose all items are between 0 and 9 as in this example:

| 7 | 0 | 4 | 0 | 9 | 1 | 9 | 1 | 9 | 5 | 3 | 7 | 3 | 1 | 6 | 7 | 4 | 2 | 0 |

| 3 | 3 | 1 | 2 | 2 | 1 | 1 | 3 | 0 | 3 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

| <0 | <1 | <2 | <3 | <4 | <5 | <6 | <7 | <8 | <9 |

| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 | 7 | 7 | 7 | 9 | 9 | 9 | 9 |

| 0 | 3 | 6 | 9 | 11 | 12 | 13 | 16 | 16 |

• “Counts” line gives # occurrences of each key.

• “Running sum” gives cumulative count of keys ≤ each value...

• …which tells us where to put each key:

• The first instance of key \( k \) goes into slot \( m \), where \( m \) is the number of key instances that are < \( k \).
Radix Sort

Idea: Sort keys one character at a time.
- Can use distribution counting for each digit.
- Can work either right to left (LSD radix sort) or left to right (MSD radix sort)
- LSD radix sort is venerable: used for punched cards.

Initial: set, cat, cad, con, bat, can, be, let, bet

Pass 1 (by char #2)  
be, cad, con, set, cat, bat, let, bet  
'_' 'd' 'n' 't'

Pass 2 (by char #1)  
cad, can, cat, bat, be, set, let, bet, con  
'_' 'a' 'e' 'o'

Pass 3 (by char #0)  
bet, can, con, bat, be, set, let, bet, con  
'_' 'b' 'c' 'l' 's'

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MSD Radix Sort

- A bit more complicated: must keep lists from each step separate
- But, can stop processing 1-element lists

<table>
<thead>
<tr>
<th>A</th>
<th>posn</th>
</tr>
</thead>
<tbody>
<tr>
<td>☆ set, cat, cad, con, bat, can, be, let, bet</td>
<td>0</td>
</tr>
<tr>
<td>☆ bat, be, bet / cat, cad, con, can / let / set</td>
<td>1</td>
</tr>
<tr>
<td>bat / ☆ be, bet / cat, cad, con, can / let / set</td>
<td>2</td>
</tr>
<tr>
<td>bat / be / bet / ☆ cat, cad, con, can / let / set</td>
<td>1</td>
</tr>
<tr>
<td>bat / be / bet / ☆ cat, cad, can / con / let / set</td>
<td>2</td>
</tr>
<tr>
<td>bat / be / bet / cad / can / cat / con / let / set</td>
<td></td>
</tr>
</tbody>
</table>
Performance of Radix Sort

• Radix sort takes $\Theta(B)$ time where $B$ is total size of the key data.
• Have measured other sorts as function of \#records.
• How to compare?
• To have $N$ different records, must have keys at least $\Theta(\lg N)$ long [why?]
• Furthermore, comparison actually takes time $\Theta(K)$ where $K$ is size of key in worst case [why?]
• So $N \lg N$ comparisons really means $N(\lg N)^2$ operations.
• While radix sort takes $B = N \lg N$ time.
• On the other hand, must work to get good constant factors with radix sort.
And Don’t Forget Search Trees

**Idea:** A search tree is in sorted order, when read in inorder.

- Need *balance* to really use for sorting [next topic].
- *Given* balance, same performance as heapsort: \( N \) insertions in time \( \lg N \) each, plus \( \Theta(N) \) to traverse, gives

\[
\Theta(N + N \lg N) = \Theta(N \lg N)
\]
Summary

• Insertion sort: $\Theta(Nk)$ comparisons and moves, where $k$ is maximum amount data is displaced from final position.
  - Good for small datasets or almost ordered data sets.
• Quicksort: $\Theta(N \lg N)$ with good constant factor if data is not pathological. Worst case $O(N^2)$.
• Merge sort: $\Theta(N \lg N)$ guaranteed. Good for external sorting.
• Heapsort, treesort with guaranteed balance: $\Theta(N \lg N)$ guaranteed.
• Radix sort, distribution sort: $\Theta(B)$ (number of bytes). Also good for external sorting.