Announcements:

- Please use bug-submit for code problems.
- Watch the newsgroup and class web site for updates, hints, useful new utilities, etc.

Readings for Today:  Data Structures (Into Java), Chapter 1;

Readings for next Topics:  Data Structures, Chapter 2-4
What Are the Questions?

• Cost is a principal concern throughout engineering:
  “An engineer is someone who can do for a dime what any fool can do for a dollar.”

• Cost can mean
  - Operational cost (for programs, time to run, space requirements).
  - Development costs: How much engineering time? When delivered?
  - Costs of failure: How robust? How safe?

• Is this program fast enough? Depends on:
  - For what purpose;
  - What input data.

• How much space (memory, disk space)?
  - Again depends on what input data.

• How will it scale, as input gets big?
Problem: Scan a text corpus (say $10^7$ bytes or so), and find and print the 20 most frequently used words, together with counts of how often they occur.

- Solution 1 (Knuth): Heavy-Duty data structures
  - Hash Trie implementation, randomized placement, pointers galore, several pages long.

- Solution 2 (Doug McIlroy): UNIX shell script:
  ```
  tr -c -s '[:alpha:]' '[\n*]' < FILE | \
  sort | \
  uniq -c | \
  sort -n -r -k 1,1 | \
  sed 20q
  ```

- Which is better?
  - #1 is much faster,
  - but #2 took 5 minutes to write and processes 20MB in 1 minute.
  - I pick #2.

- In most cases, anything will do: Keep It Simple.
Cost Measures (Time)

- Wall-clock or execution time
  - You can do this at home:
    
    \[
    \text{time java FindPrimes 1000}
    \]
  - Advantages: easy to measure, meaning is obvious.
  - Appropriate where time is critical (real-time systems, e.g.).
  - Disadvantages: applies only to specific data set, compiler, machine, etc.

- Number of times certain statements are executed:
  - Advantages: more general (not sensitive to speed of machine).
  - Disadvantages: doesn't tell you actual time, still applies only to specific data sets.

- Symbolic execution times:
  - That is, formulas for execution times or statement counts in terms of input size.
  - Advantages: applies to all inputs, makes scaling clear.
  - Disadvantage: practical formula must be approximate, may tell very little about actual time.
Asymptotic Cost

• Symbolic execution time lets us see shape of the cost function.

• Since we are approximating anyway, pointless to be precise about certain things:
  - Behavior on small inputs:
    * Can always pre-calculate some results.
    * Times for small inputs not usually important.
  - Constant factors (as in “off by factor of 2”):
    * Just changing machines causes constant-factor change.

• How to abstract away from (i.e., ignore) these things?
Handy Tool: Order Notation

- Idea: Don’t try to produce specific functions that specify size, but rather families of similar functions.

- Say something like “$f$ is bounded by $g$ if it is in $g$’s family.”

- For any function $g(x)$, the functions $2g(x)$, $1000g(x)$, or for any $K > 0$, $K \cdot g(x)$, all have the same “shape”. So put all of them into $g$’s family.

- Any function $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some constant $M$) has $g$’s shape “except for small values.” So put all of these in $g$’s family.

- If we want upper limits, throw in all functions that are everywhere $\leq$ some other member of $g$’s family. Call this family $O(g)$ or $O(g(n))$.

- Or, if we want lower limits, throw in all functions that are everywhere $\geq$ some other member of $g$’s family. Call this family $\Omega(g)$.

- Finally, define $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions bracketed by members of $g$’s family.
Big Oh

• Goal: Specify bounding from above.

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\[ M = 1 \]

Here, \( f(x) \leq 2g(x) \) as long as \( x > 1 \).

So \( f(x) \) is in \( g \)'s upper-bound family, written

\[ f(x) \in O(g(x)), \]

... even though \( f(x) > g(x) \) everywhere.
Big Omega

• Goal: Specify bounding from below:

\[ M = 1 \]

- Here, \( f'(x) \geq \frac{1}{2}g(x) \) as long as \( x > 1 \),
- So \( f'(x) \) is in \( g \)'s lower-bound family, written
  \[
  f'(x) \in \Omega(g(x)),
  \]
- ...even though \( f(x) < g(x) \) everywhere.
- In fact, we also have \( f'(x) \in O(g(x)) \) and \( f(x) \in \Omega(g(x)) \) and so we can also write
  \[
  f(x), f'(x) \in \Theta(g(x)).
  \]
Why It Matters

- Computer scientists often talk as if constant factors didn't matter at all, only the difference of \( \Theta(N) \) vs. \( \Theta(N^2) \).

- In reality they do, but we still have a point: at some point, constants get swamped.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 16 \lg n )</th>
<th>( \sqrt{n} )</th>
<th>( n )</th>
<th>( n \lg n )</th>
<th>( n^2 )</th>
<th>( n^3 )</th>
<th>( 2^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16</td>
<td>1.4</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>64</td>
<td>16</td>
</tr>
<tr>
<td>8</td>
<td>48</td>
<td>2.8</td>
<td>8</td>
<td>24</td>
<td>64</td>
<td>512</td>
<td>256</td>
</tr>
<tr>
<td>16</td>
<td>64</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>4,096</td>
<td>65,636</td>
</tr>
<tr>
<td>32</td>
<td>80</td>
<td>5.7</td>
<td>32</td>
<td>160</td>
<td>1024</td>
<td>32,768</td>
<td>4.2 \times 10^9</td>
</tr>
<tr>
<td>64</td>
<td>96</td>
<td>8</td>
<td>64</td>
<td>384</td>
<td>4,096</td>
<td>262,144</td>
<td>1.8 \times 10^{19}</td>
</tr>
<tr>
<td>128</td>
<td>112</td>
<td>11</td>
<td>128</td>
<td>896</td>
<td>16,384</td>
<td>2.1 \times 10^9</td>
<td>3.4 \times 10^{38}</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>1,024</td>
<td>160</td>
<td>32</td>
<td>1,024</td>
<td>10,240</td>
<td>1.0 \times 10^6</td>
<td>1.1 \times 10^9</td>
<td>1.8 \times 10^{308}</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( 2^{20} )</td>
<td>320</td>
<td>1024</td>
<td>( 1.0 \times 10^6 )</td>
<td>( 2.1 \times 10^7 )</td>
<td>( 1.1 \times 10^{12} )</td>
<td>( 1.2 \times 10^{18} )</td>
<td>( 6.7 \times 10^{315.652} )</td>
</tr>
</tbody>
</table>
Some Intuition on Meaning of Growth

- How big a problem can you solve in a given time?
- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size $N$.
- Entries show the size of problem that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.
- $N = \text{problem size}$

<table>
<thead>
<tr>
<th>Time ($\mu$sec) for problem size $N$</th>
<th>1 second</th>
<th>Max $N$ Possible in 1 hour</th>
<th>Max $N$ Possible in 1 month</th>
<th>Max $N$ Possible in 1 century</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lg N$</td>
<td>$10^{300000}$</td>
<td>$10^{1000000000}$</td>
<td>$10^{8} \cdot 10^{11}$</td>
<td>$10^{9} \cdot 10^{14}$</td>
</tr>
<tr>
<td>$N$</td>
<td>$10^6$</td>
<td>$3.6 \cdot 10^9$</td>
<td>$2.7 \cdot 10^{12}$</td>
<td>$3.2 \cdot 10^{15}$</td>
</tr>
<tr>
<td>$N \lg N$</td>
<td>63000</td>
<td>$1.3 \cdot 10^8$</td>
<td>$7.4 \cdot 10^{10}$</td>
<td>$6.9 \cdot 10^{13}$</td>
</tr>
<tr>
<td>$N^2$</td>
<td>1000</td>
<td>60000</td>
<td>1.6 $\cdot 10^6$</td>
<td>5.6 $\cdot 10^7$</td>
</tr>
<tr>
<td>$N^3$</td>
<td>100</td>
<td>1500</td>
<td>14000</td>
<td>150000</td>
</tr>
<tr>
<td>$2^N$</td>
<td>20</td>
<td>32</td>
<td>41</td>
<td>51</td>
</tr>
</tbody>
</table>
Using the Notation

• Can use this order notation for any kind of real-valued function.

• We will use them to describe cost functions. Example:

```java
/** Find position of X in list L. Return -1 if not found */
int find (List L, Object X) {
    int c;
    for (c = 0; L != null; L = L.next, c += 1)
        if (X.equals (L.head)) return c;
    return -1;
}
```

• Choose representative operation: number of `.equals` tests.

• If $N$ is length of $L$, then loop does at most $N$ tests: worst-case time is $N$ tests.

• In fact, total # of instructions executed is roughly proportional to $N$ in the worst case, so can also say worst-case time is $O(N)$, regardless of units used to measure.

• Use $N > M$ provision (in defn. of $O(\cdot)$) to handle empty list.
Careful!

- It’s also true that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ also: Big-Oh bounds are loose.
- The worst-case time is $\Omega(N)$, since $N \in \Omega(N)$, but that does not mean that the loop always takes time $N$, or even $K \cdot N$ for some $K$.
- Instead, we are just saying something about the function that maps $N$ into the largest possible time required to process an array of length $N$.
- To say as much as possible about our worst-case time, we should try to give a $\Theta$ bound: in this case, we can: $\Theta(N)$.
- But again, that still tells us nothing about best-case time, which happens when we find $x$ at the beginning of the loop. Best-case time is $\Theta(1)$.
Effect of Nested Loops

- Nested loops often lead to polynomial bounds:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = 0; j < A.length; j += 1)
        if (i != j && A[i] == A[j])
            return true;
return false;
```

- Clearly, time is $O(N^2)$, where $N = A.length$. Worst-case time is $\Theta(N^2)$.

- Loop is inefficient though:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = i+1; j < A.length; j += 1)
        if (A[i] == A[j]) return true;
return false;
```

- Now worst-case time is proportional to

$$N - 1 + N - 2 + \ldots + 1 = N(N - 1)/2 \in \Theta(N^2)$$

(so asymptotic time unchanged by the constant factor).
Recursion and Recurrences: Fast Growth

• Silly example of recursion:

```java
/** True iff X is a substring of S */
boolean occurs (String S, String X) {
    if (S.equals (X)) return true;
    if (S.length () <= X.length ()) return false;
    return
        occurs (S.substring (1), X) ||
        occurs (S.substring (0, S.length ()-1), X);
}
```

• In the worst case, both recursive calls happen.

• Define $C(N)$ to be the worst-case cost of `occurs(S,X)` for $S$ of length $N$, $X$ of fixed size $N_0$, measured in # of calls to `occurs`. Then

$$C(N) = \begin{cases} 
1, & \text{if } N \leq N_0, \\
2C(N-1) + 1 & \text{if } N > N_0
\end{cases}$$

• So $C(N)$ grows exponentially:

$$C(N) = 2C(N-1) + 1 = 2(2C(N-2) + 1) + 1 = \ldots = 2(\ldots 2 \cdot 1 + 1) + \ldots + 1 = 2^{N-N_0} - 1 \in \Theta(2^N)$$
**Binary Search: Slow Growth**

```java
/** True X iff is an element of S[L .. U]. Assumes
 *  S in ascending order, 0 <= L <= U-1 < S.length. */
boolean isIn (String X, String[] S, int L, int U) {
    if (L > U) return false;
    int M = (L+U)/2;
    int direct = X.compareTo (S[M]);
    if (direct < 0) return isIn (X, S, L, M-1);
    else if (direct > 0) return isIn (X, S, M+1, U);
    else return true;
}
```

- Here, worst-case time, $C(D)$, (as measured by # of string comparisons), depends on size $D = U - L + 1$.
- We eliminate $S[M]$ from consideration each time and look at half the rest. Assume $D = 2^k - 1$ for simplicity, so:

$$C(D) = \begin{cases} 
0, & \text{if } D \leq 0, \\
1 + C((D - 1)/2), & \text{if } D > 0.
\end{cases}$$

$$= 1 + 1 + \ldots + 1 + 0$$

$$= k = \lceil \log D \rceil \in \Theta(\log D)$$
Another Typical Pattern: Merge Sort

List sort (List L) {
    if (L.length () < 2) return L;
    Split L into L0 and L1 of about equal size;
    L0 = sort (L0); L1 = sort (L1);
    return Merge of L0 and L1
}

Merge ("combine into a single, ordered list") takes time proportional to size of its result.

• Assuming that size of L is \( N = 2^k \), worst-case cost function, \( C(N) \), counting just merge time (\( \propto \# \) items merged):

\[
C(N) = \begin{cases} 
1, & \text{if } N < 2; \\
2C(N/2) + N, & \text{if } N \geq 2.
\end{cases}
\]

\[
= 2(2C(N/4) + N/2) + N
\]

\[
= 4C(N/4) + N + N
\]

\[
= 8C(N/8) + N + N + N
\]

\[
= N \cdot 1 + \underbrace{N + N + \ldots + N}_{k=\lg N}
\]

\[
= N + N \lg N \in \Theta(N \lg N)
\]

• In general, \( \Theta(N \lg N) \) for arbitrary \( N \) (not just \( 2^k \)).
Amortization: Expanding Vectors

• When using array for expanding sequence, best to double size of array to grow it. Here’s why.

• If array is size $s$, doubling its size and moving $s$ elements to the new array takes time $\propto 2s$.

• Cost of inserting $N$ items into array, doubling size as needed, starting with array size 1:

<table>
<thead>
<tr>
<th>To Insert Item #</th>
<th>Resizing Cost</th>
<th>Cumulative Cost</th>
<th>Resizing Cost per Item</th>
<th>Array Size After Insertions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3 to 4</td>
<td>4</td>
<td>6</td>
<td>1.5</td>
<td>4</td>
</tr>
<tr>
<td>5 to 8</td>
<td>8</td>
<td>14</td>
<td>1.75</td>
<td>8</td>
</tr>
<tr>
<td>$2^m + 1$ to $2^{m+1}$</td>
<td>$2^m+1$</td>
<td>$2^{m+2} - 2$</td>
<td>$\approx 2$</td>
<td>$2^{m+1}$</td>
</tr>
</tbody>
</table>

• If we spread out (amortize) the cost of resizing, we average about 2 time units on each item: “amortized insertion time is 2 units.”

• So even though worst-case time for adding one element to array of $N$ elements is $2N$, time to add $N$ elements is $\Theta(N)$, not $\Theta(N^2)$. 
Demonstrating Amortized Time: Potential Method

- To formalize the argument, associate a potential, \( \Phi_i \geq 0 \), to the \( i \)\(^{th} \) operation that keeps track of “saved up” time from cheap operations that we can “spend” on later expensive ones. Start with \( \Phi_0 = 0 \).

- Now define the amortized cost of the \( i \)\(^{th} \) operation as
  \[
  a_i = c_i + \Phi_{i+1} - \Phi_i,
  \]
  where \( c_i \) is the real cost of the operation.

- On cheap operations, we artificially set \( a_i > c_i \) and increase \( \Phi \) (\( \Phi_{i+1} > \Phi_i \)).

- On expensive ones, we typically have \( a_i \ll c_i \) and greatly decrease \( \Phi \) (but don’t let it go negative—may not be “overdrawn”).

- We try to do all this so that \( a_i \) remains as we desired (e.g., \( O(1) \) for expanding array), without allowing \( \Phi_i < 0 \).

- Requires that we choose \( a_i \) so that \( \Phi_i \) always stays ahead of \( c_i \).
Application to Expanding Arrays

• When adding to our array, the cost, $c_i$, of adding element $#i$ when the array already has space for it is 1 unit.

• The array does not initially have space when adding items 1, 2, 4, 8, 16,... — in other words at item $2^n$ for all $n \geq 0$. So,
  
  $- c_i = 1$ if $i \geq 0$ and is not a power of 2; and
  $- c_i = 3i + 1$ (allocate $2i$ items, copy $i$ items, and then add item $#i$) when $i$ is a power of 2.

• So on each operation $#2^n$ we’re going to need to have saved up at least $3 \cdot 2^n$ units of potential to cover the expense, and we have the preceding $2^{n-1}$ operations to do it (the ones since the preceding doubling operation).

• To do so, just choose $a_i = 7$ (or could let $a_0 = 1, a_1 = 4$)

• Here’s what happens:

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_i$</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>1</td>
<td>13</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>25</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>49</td>
</tr>
<tr>
<td>$a_i$</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
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<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$\Phi_i$</td>
<td>0</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>15</td>
<td>9</td>
<td>15</td>
<td>21</td>
<td>27</td>
<td>9</td>
<td>15</td>
<td>21</td>
<td>27</td>
<td>33</td>
<td>39</td>
<td>45</td>
<td>51</td>
</tr>
</tbody>
</table>