1. (2 pts.) Getting started
What is Anant Sahai’s second favorite number?
The answer is found on Piazza.
(Why are we having you do this? Piazza is your best source for recent announcements, clarifications on
homeworks, and related matters, and we want you to be familiar with how to read the newsgroup.)

**Answer:** Anant’s second favorite number is \( \ln 2 \) or the natural logarithm of 2. Hopefully you all got this
by looking at Piazza, which is going to be your main source of outside-the-class communication with the
course staff.

2. (2 pts.) Tricksy profs
Prof. Sahai and his brother (who in real life is a CS Prof at UCLA) are guarding a pair of magical doors that
appear to be identical. But one of the doors leads you to a room full of Smaug’s treasure and the other to a
room full of angry goats. You want to get to the room filled with treasure. Both profs know which door is
which as well as knowing each other.

The problem is that one of them always lies and the other always tells the truth. But you don’t know which
is which. Can you ask a question whose answer will reveal to you which door to take? What question should
this be? Argue why this works or why no question could possibly work.

**Answer:** There are many ways to solve this problem. One possible way is to ask one of the two profs this
question: "If I asked the other person whether the door you are guarding leads to the treasure, would he
answer yes?". Now if you negate the answer you get you would obtain the true answer to the question "Does
the door next to the person I asked my question from lead to the treasure?". This is because if you are asking
the liar, he would negate his answer; but even when you ask the truth-teller he would tell you the truth, but
the truth is already negated once because you are asking him what the other person (the liar) would say.

An easier way to check this is to use the following table which explores all possibilities.

<table>
<thead>
<tr>
<th>Person</th>
<th>His door leads to</th>
<th>Other person</th>
<th>Correct answer to your question</th>
<th>What you hear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liar</td>
<td>Goats</td>
<td>Truth-teller</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Liar</td>
<td>Treasure</td>
<td>Truth-teller</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Truth-teller</td>
<td>Goats</td>
<td>Liar</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Truth-teller</td>
<td>Treasure</td>
<td>Liar</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

So as you can see the answer you hear determines which door leads to the treasure.

3. (2 pts.) Exclusive OR
The "exclusive OR" connective (written as XOR or \( \oplus \)) is just what it sounds like: \( P \oplus Q \) is true when exactly
one of \( P, Q \) is true (but not both). Write down the truth table for \( P \oplus Q \) and for \((P \land \neg Q) \lor (\neg P \land Q)\), and
hence show that \( P \oplus Q \) is logically equivalent to \((P \land \neg Q) \lor (\neg P \land Q)\).

**Answer:** As you can see from the following truth table, the columns for \( P \oplus Q \) and \((P \land \neg Q) \lor (\neg P \land Q)\) are
exactly the same, and thus these two expressions are logically equivalent.
4. (8 pts.) Implications

Which of the following implications is true?

1. If 30 is divisible by 10 then 40 is divisible by 10. Answer: True. The conclusion is true.
2. If 30 is divisible by 9 then 40 is divisible by 10. Answer: True. The premise is false.
3. If 30 is divisible by 10 then 40 is divisible by 9. Answer: False. The premise is true but the conclusion is false.
4. If 30 is divisible by 9 then 40 is divisible by 9. Answer: True. The premise is false.

5. (2 pts.) Chasing chains of consequences

Suppose that $S$ is a set of integers with the following properties:

1. For all $x$, if $x \in S$, then $-x \in S$.
2. For all $x$, if $x \in S$, then $2x \in S$.

Suppose that the numbers 10, 11, and 12 are not in $S$. What can you say about $S$? Tell us as much as you can.

Answer: We will use the contrapositives of the two if statements which are logically equivalent.

1. For all $x$, if $-x \notin S$, then $x \notin S$.
2. For all $x$, if $2x \notin S$, then $x \notin S$.

Applying the second rule for $x = 5$ tells us that $5 \notin S$ because $10 \notin S$. Applying it for $x = 6$ tells us that $6 \notin S$, because $12 \notin S$. Applying this rule one more time for $x = 3$ tells us that $3 \notin S$ because $6 \notin S$.

So far we know that $3, 5, 6, 10, 11, 12 \notin S$. Now applying the first rule for $x \in \{-3, -5, -6, -10, -11, -12\}$, we can deduce that $\{\pm 3, \pm 5, \pm 6, \pm 10, \pm 11, \pm 12\} \notin S$. A quick check tells us that we can no longer apply any of the rules to get new numbers.

Now we will argue that we can’t know any other fact about the membership of other numbers in $S$. First note that $S = \emptyset$ (the empty set) satisfies our two rules. Therefore it is possible for any number not to be in $S$.

For the second example let $S$ be the set of all natural numbers except $\{\pm 3, \pm 5, \pm 6, \pm 10, \pm 11, \pm 12\}$. This set satisfies our two rules, because for any $x$, if $2x \notin S$ then $2x \in \{\pm 3, \pm 5, \pm 6, \pm 10, \pm 11, \pm 12\}$ and then it is easy to check that $x \in \{\pm 3, \pm 5, \pm 6, \pm 10, \pm 11, \pm 12\}$, therefore $x \notin S$. A similar argument can be made for the other rule.

For any number $y \notin \{\pm 3, \pm 5, \pm 6, \pm 10, \pm 11, \pm 12\}$, $y$ is present in our second example and it is not present in our first example. Therefore we cannot know anything about the membership of $y$ in $S$.

6. (10 pts.) Practice with quantifiers

Which of the following propositions is true? In part 5, $Q(k)$ denotes the proposition "$1 + 2 + \cdots + k = k(k+1)/2$".
1. \((\forall x \in \mathbb{N}. x^2 < 5) \implies (\forall x \in \mathbb{N}. x^2 < 4)\). **Answer:** True. The premise which is \((\forall x \in \mathbb{N}. x^2 < 5)\) is false \((x = 3)\) is a counterexample). Therefore the implication is true.

2. \((\forall x \in \mathbb{N}. x^2 < 4) \implies (\forall x \in \mathbb{N}. x^2 < 5)\). **Answer:** True. Again the premise which is \((\forall x \in \mathbb{N}. x^2 < 4)\) is false \((x = 2)\) is a counterexample). Therefore the implication is true.

3. \(\forall x \in \mathbb{N}. (x^2 < 5 \implies x^2 < 4)\). **Answer:** False. For \(x = 2\), we have \(x^2 < 5\) is true, but \(x^2 < 4\) is false. Therefore for \(x = 2\) the implication \(x^2 < 5 \implies x^2 < 4\) is false.

4. \(\forall x \in \mathbb{N}. (x^2 < 4 \implies x^2 < 5)\). **Answer:** True. For any arbitrary \(x \in \mathbb{N}\), either \(x^2 < 4\) or \(x^2 \geq 4\). In the first case we have \(x^2 < 4 < 5\) which means that the conclusion in the implication is true, and in the second case the premise is false. So either way the implication is true.

5. \(\forall n \in \mathbb{N}. Q(n) \implies Q(n + 1)\). **Answer:** True. To prove the inside implication for an arbitrary \(n\) we can follow these steps: if \(Q(n)\) is false, there is nothing to prove since the implication is by default true. So assume that \(Q(n)\) is true which means that \(1 + \cdots + n = n(n + 1)/2\). Now adding \(n + 1\) to both sides gives us

\[
1 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1) = \frac{(n + 1)(n + 2)}{2}.
\]

But this is exactly \(Q(n + 1)\). So in this case the conclusion is true and hence the implication is true. So we showed that for any arbitrary \(n\), \(Q(n) \implies Q(n + 1)\) is true.

7. **(4 pts.) Working with quantifiers**

Recall that \(\mathbb{N} = 0, 1, 2, \ldots\) denotes the set of natural numbers.

1. For any natural number \(n\), let \(P(n)\) denote the proposition

\[
P(n) = \forall i \in \mathbb{N}. (i < n \implies (\forall j \in \mathbb{N}. (j = n \lor n \neq ij))).
\]

Concisely, for which numbers \(n \in \mathbb{N}\) is \(P(n)\) true?

**Answer:** To decipher what \(P(n)\) is really saying, we will work our way from the inside out. The expression \(\forall j \in \mathbb{N}. (j = n \lor n \neq ij)\) is saying that \(n\) is not a multiple of \(i\) (note that \(ij\) ranges over all multiples of \(i\)) unless the multiplication factor is \(n\) (in which case \(i\) has to be 1 or \(n\) and \(i\) would be both 0).

The implication \(i < n \implies (\forall j \in \mathbb{N}. (j = n \lor n \neq ij))\) is false when \(i < n\) and the conclusion is false. Therefore it is false when \(i < n\) and \(i \neq 1\) and \(n\) is divisible by \(i\) (in the special case where \(n = i = 0, j = 1\) is a counterexample to the conclusion of the implication which means that the conclusion is still false).

Therefore \(P(n)\) is true when \(n\) has no divisor \(i\) for which \(i \neq 1\) and \(i < n\). This is true for all prime numbers, 0, and 1. A prime number never has a divisor other than 1 and itself. The number 1 has only itself as a divisor. The number 0 has every number as its divisor but none of them are less than itself. Therefore every prime number, 0, and 1 has no divisor other than 1 which is less than itself. On the other side any composite number \(n > 1\) can be written as \(ij\) where \(1 < i, j < n\). Therefore for every composite number \(P(n)\) is false.

2. Rewrite the following quantified proposition in an equivalent form with all negations ("\(\lnot\", ", "\neq\)"") removed.

\[
\neg \forall i \in \mathbb{N}. \neg \exists j \in \mathbb{N}. \exists k \in \mathbb{N}. \forall \ell \in \mathbb{N}. f(i, j) \neq g(k, \ell).
\]
Answer: We will move the negations inwards and use the rules stated in the lecture (we change the quantifiers).

\[ \neg \forall i \in \mathbb{N}. \forall j \in \mathbb{N}. \exists k \in \mathbb{N}. \neg \forall \ell \in \mathbb{N}. f(i, j) \neq g(k, \ell) \equiv \]
\[ \neg \forall i \in \mathbb{N}. \exists j \in \mathbb{N}. \exists k \in \mathbb{N}. \neg \forall \ell \in \mathbb{N}. f(i, j) = g(k, \ell) \equiv \]
\[ \neg \forall i \in \mathbb{N}. \forall j \in \mathbb{N}. \neg \exists k \in \mathbb{N}. \exists \ell \in \mathbb{N}. f(i, j) = g(k, \ell) \equiv \]
\[ \neg \forall i \in \mathbb{N}. \forall j \in \mathbb{N}. \forall k \in \mathbb{N}. \forall \ell \in \mathbb{N}. f(i, j) \neq g(k, \ell) \equiv \]
\[ \exists i \in \mathbb{N}. \forall j \in \mathbb{N}. \forall k \in \mathbb{N}. \forall \ell \in \mathbb{N}. f(i, j) \neq g(k, \ell) \equiv \]
\[ \exists i \in \mathbb{N}. \exists j \in \mathbb{N}. \exists k \in \mathbb{N}. \exists \ell \in \mathbb{N}. f(i, j) = g(k, \ell). \]

An easier way would have been to look at each quantifier and count the number of negations in front of it. For an odd number of negations we should switch the quantifier and for an even number we should leave it alone, as each negation that passes a quantifier flips it once.

8. (8 pts.) A few proofs

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in the Lecture Notes) you used.

1. For all natural numbers \( n \), if \( n^2 \) is even then \( n^5 \) is even.

   Answer: To show you that you can always use multiple techniques to prove something, we will provide two different methods here.

   First, we will use a direct proof. Assume that \( n^2 \) is even. Then there exists an integer \( k \) such that \( n^2 = 2k \). Then \( n^5 = n^2 \times n^3 = 2(kn^3) \) which is twice an integer. Therefore \( n^5 \) is also even.

   Second, we will prove this statement by cases. For any generic natural number \( n \) there are two cases.

   (a) \( n \) is even. In this case \( n^5 \) is a multiple of \( n \) which is even, so the conclusion of the implication is true. Hence the implication is true.

   (b) \( n \) is odd. The product of two odd numbers is odd. Therefore \( n^2 \) is odd. This means that the premise of the implication is false. Hence the implication is true.

2. For all natural numbers \( n \), \( n^2 - n + 3 \) is odd.

   Answer: We will prove this statement by cases. For any generic natural number \( n \) there are two cases.

   (a) \( n \) is even. In this case \( n^2 \) is even and \( n \) is even. The difference between even numbers is even and therefore \( n^2 - n \) is even. The sum of an even number and an odd number is odd, therefore \( n^2 - n + 3 \) is odd.

   (b) \( n \) is odd. In this case \( n^2 \) is odd and \( n \) is odd. The difference between odd numbers is even and therefore \( n^2 - n \) is even. The sum of an even number and an odd number is odd, therefore \( n^2 - n + 3 \) is odd.

3. For all real numbers \( x, y \), if \( x + y \geq 20 \) then \( x \geq 10 \) or \( y \geq 10 \).

   Answer: We will prove this statement by contraposition. For generic \( x \) and \( y \) we assume the negation of the conclusion, which is that \( x < 10 \) and \( y < 10 \). Now adding these two inequalities together we get \( x + y < 20 \) which is the negated premise. Therefore from the negated conclusion we derived the negated premise. Hence, the contrapositive is true.
4. For all real numbers \( r \), if \( r \) is irrational then \( r^2 \) is irrational.

**Answer:** We will disprove this statement by proving its negative. According to the rules of negation for quantifiers, the negative becomes: there exists a natural number \( r \), such that \( r \) is irrational and \( r^2 \) is rational. To prove an existential statement, it is enough to provide an example, which is essentially a counterexample to the original proposition. This is a direct proof (of the negative), which might also be called a disproof of the original proposition by counterexample.

There are plenty of examples, one of which is \( r = \sqrt{2} \). It is well-known that \( \sqrt{2} \) is irrational, whereas \( r^2 = 2 \) is clearly rational.

9. (2 pts.) You be the grader

Assign a grade of A (correct) or F (failure) to the following proof. If you give an F, please explain exactly everything that is wrong with the structure or the reasoning in the proof. Justify your answer (saying the claim is false is not a justification).

**Theorem 0.1:** \( \forall n \in \mathbb{N}. n^2 \leq n \implies (n + 1)^2 \leq n + 1. \)

**Proof:** Suppose that \( n \in \mathbb{N} \) and \( n^2 \leq n \). (Otherwise, there is nothing to prove.) We need to show that \( (n + 1)^2 \leq n + 1 \).

Working backwards we see that:

\[
\begin{align*}
(n + 1)^2 & \leq n + 1 \\
n^2 + 2n + 1 & \leq n + 1 \\
n^2 + 2n & \leq n \\
n^2 & \leq n
\end{align*}
\]

So we get back to our original hypothesis which was assumed to be true. Hence, for every \( n \in \mathbb{N} \) we know that if \( n^2 \leq n \), then \( (n + 1)^2 \leq n + 1 \).

**Answer:** This is a classic F! The proof only proves the converse of our original implication (which is not logically equivalent to the implication). In other words, we assume what we are trying to prove in the beginning, which is the source of failure.

There is also another problem with the proof. In a mathematical proof, every step needs to be sufficiently justified. Anyone with a reasonable background in math should be able to follow the proof. Here in this proof, the step going from \( n^2 + 2n \leq n \) to \( n^2 \leq n \), while mathematically justifiable, is not sufficiently documented. In this step we are removing some nonnegative amount \( 2n \) from the smaller side of the inequality, which is mathematically valid. But the proof should have mentioned the fact that \( 2n \geq 0 \) because \( n \in \mathbb{N} \).