1. (4 pts.) Proof by induction
For \( n \in \mathbb{N} \) with \( n \geq 2 \), define \( s_n \) by
\[
    s_n = \left( 1 - \frac{1}{2} \right) \times \left( 1 - \frac{1}{3} \right) \times \cdots \times \left( 1 - \frac{1}{n} \right).
\]
Prove that \( s_n = 1/n \) for every natural number \( n \geq 2 \).

**Answer:** We will use induction on \( n \).

- **Base case:** \( s_2 = 1 - \frac{1}{2} = \frac{1}{2} \).
- **Inductive hypothesis:** Suppose \( n \geq 2 \). We may assume \( s_n = \frac{1}{n} \).
- **Inductive step:** We have
\[
    s_{n+1} = \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{1}{3} \right) \cdots \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{1}{n+1} \right)
    = s_n \times \left( 1 - \frac{1}{n+1} \right)
    = \frac{1}{n} \times \frac{n}{n+1}
    = \frac{1}{n+1},
\]
which is what we wanted to show.

2. (4 pts.) Another induction proof
Let \( a_n = 3^{n+2} + 2^{2n+1} \). Prove that 13 divides \( a_n \) for every \( n \in \mathbb{N} \).
(Hint: What can you say about \( a_{n+1} - 3a_n \)?)

**Answer:** We induct on \( n \).

- **Base case:** \( a_0 = 3^2 + 4^1 = 13 \).
- **Inductive hypothesis:** Assume \( P(n) \) is true. In other words, we may assume \( 13 \mid a_n \).
- **Inductive step:** We compute
\[
    a_{n+1} - 3a_n = 3^{(n+1)+2} + 2^{2(n+1)+1} - 3(3^{n+2} + 2^{2n+1})
    = 3^{n+3} + 2^{2n+3} - 3^{n+3} - 3 \cdot 2^{2n+1}
    = 2 \cdot 4^{2n+1} - 3 \cdot 2^{2n+1}
    = 13 \cdot 2^{2n+1},
\]
so \( a_{n+1} = 3a_n + 13 \cdot 2^{2n+1} \). By the inductive hypothesis \( 13 \mid a_n \). Since \( 13 \mid 13 \cdot 2^{2n+1} \), the RHS (Right-Hand Side) is divisible by 13. Then so is the LHS (Left-Hand Side), proving the claim.
Inductive step (another approach): We compute

\[
a_{n+1} = 3^{(n+1)+2} + 4^{2(n+1)+1} \\
= 3^{n+3} + 4^{2n+3} \\
= 3 \cdot 3^{n+2} + 16 \cdot 4^{2n+1} \\
= 3 \cdot 3^{n+2} + 3 \cdot 4^{2n+1} + 13 \cdot 4^{2n+1} \\
= 3a_n + 13 \cdot 4^{2n+1}.
\]

By the inductive hypothesis \(13|a_n\). Since \(13|13 \cdot 4^{2n+1}\), the RHS is divisible by 13. Then so is the LHS, proving the claim.

3. (4 pts.) Tower of Brahma

This puzzle was invented by the French mathematician, Edouard Lucas, in 1883. Accompanying the puzzle is a story:

In the great temple at Benares beneath the dome which marks the center of the world, rests a brass plate and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and Night unceasingly, the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When all the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple and priests alike will crumble into dust, and with a thunderclap the world will vanish.

Prove by induction the exact number of moves required to carry out this task if we assume that the priests are trying to hasten the end of the world. If there are \(n\) disks on the original needle. Assuming that the priests can move a disk each second, roughly how many centuries does the prophecy predict before the destruction of the World?

**Answer:** Let \(a_n\) denote the minimum number of moves needed to complete this task if there are \(n\) disks on the original needle. We claim that \(a_n = 2^n - 1\). To prove this we will induct on \(n\).

**Base case:** When \(n = 1\), we only need 1 move to complete the task. This agrees with our theorem, since \(2^1 - 1 = 1\).

**Inductive hypothesis:** Assume \(a_n = 2^n - 1\).

**Inductive step:** Suppose we have \(n + 1\) disks. Label the three needles \(A, B\) and \(C\), and suppose the disks start on \(A\). We solve the puzzle as follows: first, we move the \(n\) smallest disks to \(B\). Then we move the largest disk from \(A\) to \(C\). Finally, we move the \(n\) smallest disks from \(B\) to \(C\). These steps take \(a_n\), 1, and \(a_n\) moves, respectively, so using this strategy we can always complete the task in \(a_n + 1 + a_n\) moves. Thus \(a_{n+1} \leq 2a_n + 1\). By the inductive hypothesis \(a_n = 2^n - 1\), so \(a_{n+1} \leq 2(2^n - 1) + 1 = 2^{n+1} - 1\).

While we won’t require you to prove optimality, here’s how it can be proved. Consider a sequence of \(M\) moves which completes the task. Suppose the largest disk is moved for the first time on move \(i\), and is moved for the last time on move \(j\). Then \(i \leq j\) (note that we might have \(i = j\)). We may assume without loss of generality that the largest disk was moved from needle \(A\) to needle \(B\) on move \(i\) (the case where it is moved from \(A\) to \(C\) is symmetrical). This means that just before move \(i\), needle \(A\) must have contained only the largest disk, and needle \(B\) must have been empty. Therefore, at move \(i\), the \(n\) smallest disks must all be on \(C\), so at least \(a_n\) moves have already occurred. Thus \(i > a_n\). Similarly, after the \(j^{th}\) move the largest
disk is on one needle, and the other $n$ disks are on some other needle. These $n$ disks must all be moved on
top of the largest disk, so at least $a_n$ more moves are needed. Thus $a_i < i \leq j$ and $j + a_n \leq M$. Putting this
together, we see $2a_n + 1 \leq M$. So any successful sequence must be at least $2a_n + 1 = 2^{n+1} - 1$ moves long.
Then $2^{n+1} - 1 \leq a_{n+1}$. Since we have already shown $a_{n+1} \leq 2^{n+1} - 1$, we get $a_{n+1} = 2^{n+1} - 1$, proving the
claim.

Assuming that the priests can move a disk each second, roughly how many centuries does the prophecy
predict before the destruction of the World?

The prophecy predicts

$$\frac{2^{64} - 1}{60 \times 60 \times 24 \times 365 \times 100} = 5,849,420,000$$

centuries before the destruction of the World. Since geologists predict that the Earth is 45,400,000 centuries
old, and astronomers estimate the universe is 137,000,000 centuries old, we’ve still got some time left.

4. (4 pts.) The proof of the $\pi$ is in the eating

Philip J. Fry has a job at the local pizza parlor, where he tends to be a bit distractable. One day, he has a
stack of unbaked pizza doughs and for some unknown reason, he decides to arrange them in order of size,
with the largest pizza on the bottom, the next largest pizza just above that, and so on. He has learned how
to place his spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their
order). The figure below shows two sample flips.

This is the only move Fry can do to change the order of the stack; however, he is willing to keep repeating
this move until he gets the stack in order. Is it always possible for him to get the pizzas in order via some
sequence of moves, no matter how many pizzas he starts with and what order they are originally in? Prove
your answer.

**Answer: Theorem:** Given $n$ pizzas, Fry can always order them using some finite sequence of moves.

**Proof:** We induct on $n$.

**Base case:** When $n = 1$ there is nothing to prove, since the pizzas are already ordered.

**Inductive hypothesis:** We may assume that $P(n)$ is true. That is, we assume Fry can put $n$ pizzas into order.

**Inductive step:** Suppose Fry has $n + 1$ pizzas, and suppose the largest pizza is the $k^{th}$ pizza from the top.
First, Fry flips the top $k$ pizzas. This brings the largest pizza to the top of the stack. Then Fry flips the
entire stack, bringing the largest pizza to the bottom. By the inductive hypothesis, he can then sort the
other $n$ pizzas on top of the largest one, without disturbing the largest one on the bottom. Then the stack is
completely sorted.

5. (4 pts.) Boring Birds

In a casual game that didn’t quite sweep the nation, there is a bucket that contains some number of yellow
birds, red birds, and black birds. When it is a player’s turn, the player may either: (i) slingshot away one
yellow bird from the bucket, and add up to 3 red birds into the bucket; (ii) slingshot away two red birds from the bucket, and add up to 7 black birds into the bucket; or, (iii) explode a single black bird in the bucket. These are the only legal moves. The last player that can make a legal move wins.

Prove by induction that, if the bucket initially contains a finite number of birds at the start of the game, then the game will end after a finite number of moves.

**Answer:** The idea here is to assign weights 100, 10, and 1 for yellow, red, and black birds, respectively. If there are \( y \) yellow birds, \( r \) red birds, and \( b \) black birds \( (y, r, b \in \mathbb{N}) \), we can define \( v = 100y + 10r + b \) and observe the value of \( v \) will decrease by at least 1 after every legal move.

**Lemma:** The value of \( v \) will decrease by at least 1 after every legal move.

**Proof:** We can prove this by cases:

- Move (i): the value of \( v \) will change from \((100y + 10r + b)\) to \((100(y - 1) + 10(r + x) + b)\) where \(0 \leq x \leq 3\). Therefore, it will decrease by \((100y + 10r + b) - (100(y - 1) + 10(r + x) + b) = 100 - 10x\), which is \(\geq 1\) for the relevant values of \(x\).
- Move (ii): the value of \( v \) will change from \((100y + 10r + b)\) to \((100y + 10(r - 2) + (b + x))\) where \(0 \leq x \leq 7\). Therefore, it will decrease by \((100y + 10r + b) - (100y + 10(r - 2) + (b + x)) = 20 - x\), which is \(\geq 1\) for the relevant values of \(x\).
- Move (iii): the value of \( v \) will change from \((100y + 10r + b)\) to \((100y + 10r + (b - 1))\). Therefore, it will decrease by \((100y + 10r + b) - (100y + 10r + (b - 1)) = 1\).

As a result, the value of \( v \) will decrease by at least 1 after every legal move. \(\Box\)

Since \( v \) is never negative, we can’t play this game forever. This argument is logically correct, but we must formalize it through induction.

**Theorem:** Let \( v_0 \) be the value of the initial configuration and \( v_n \) be the value of the configuration after \( n \) moves where \( n \) is a natural number. Then \( v_n \leq v_0 - n \).

**Proof:** Proof by simple induction over \( n \), the number of legal moves made.

**Base case:** Let \( n = 0 \), then \( v_0 - n \) becomes \( v_0 \) and \( v_n \) becomes \( v_0 \). Trivially we see that \( v_0 \leq v_0 \).

**Inductive hypothesis:** Assume that for some \( n \in \mathbb{N} \), \( v_n \leq v_0 - n \).

**Inductive step:** Now consider the \( n + 1 \)-th turn. By the lemma, we know that the step will decrease the value of \( v_n \) by at least 1, we conclude that \( v_{n+1} \leq v_n - 1 \). Hence \( v_{n+1} \leq v_n - 1 \leq (v_0 - n) - 1 = v_0 - (n + 1) \), where in the second step we used the inductive hypothesis. The theorem follows by induction. \(\Box\)

Since \( v_n \leq v_0 - n \), we know \( n \leq v_0 - v_n \). Since \( v_n \geq 0 \) by construction (the value can never be negative since it is the sum of natural numbers multiplied by other natural numbers), we can conclude \( n \leq v_0 \). Since \( v_0 \) is clearly finite, this means that the game can have only a finite number of legal moves played and must therefore end.

In the solution above we assign weights 100, 10, and 1 for yellow, red, and black birds, respectively. However, there are other weight assignments that make this proof idea work. Another workable assignment would be to weight each yellow bird at 13, red bird at 4, and black bird at 1, i.e., \( v = 13y + 4r + b \), since that also ensures that each legal move reduces the value of \( v \) by at least 1. On the other hand, weighting every yellow bird at 1 would not work, because some moves would increase the value of \( v \), violating the intended invariant.

**What you should get out of this problem:** The main idea is to associate each state of the game with a number in a way that lets us establish a useful invariant. This is a common strategy in solving these kinds
of problems, but it need not be the only way. Notice also that when saying what you’re inducting over, you should provide more detail than just saying “We induct over k” or “Proof by induction over n”. Unless it is very clear what n means, specify what quantity n represents.

6. (16 pts.) You be the grader
Assign a grade of A (correct) or F (failure) to the following proofs. If you give a F, please explain clearly where the logical error in the proof lies. Saying that the claim is false is not a valid explanation of what is wrong with the proof. If you give an A, you do not need to explain your grade.

(a) Claim: For every \( n \in \mathbb{N} \), \( n^2 + 3n \) is odd.
   Proof: The proof will be by induction on \( n \).
   Base case: The number \( n = 1 \) is odd.
   Induction step: Suppose \( k \in \mathbb{N} \) and \( k^2 + 3k \) is odd. Then,
   \[
   (k + 1)^2 + 3(k + 1) = (k^2 + 2k + 1) + (3k + 3) = (k^2 + 3k) + (2k + 4)
   
   
   
   is the sum of an odd and an even integer. Therefore, \( (k + 1)^2 + 3(k + 1) \) is odd. Therefore, by the principle of mathematical induction, \( n^2 + 3n \) is odd for all natural numbers \( n \).

   Answer: This proof receives an A. The reasoning is valid.

(b) Claim: For every real number \( x \), if \( x \) is irrational, then \( 2008x \) is irrational.
   Proof: Suppose \( 2008x \) is rational. Then \( 2008x = \frac{p}{q} \) for some integers \( p, q \) with \( q \neq 0 \). Therefore \( x = \frac{p}{(2008q)} \) where \( p \) and \( 2008q \) are integers with \( 2008q \neq 0 \), so \( x \) is rational. Therefore, if \( 2008x \) is rational, then \( x \) is irrational. By the contrapositive, if \( x \) is irrational, then \( 2008x \) is irrational.

   Answer: This proof receives an A. The reasoning is valid.

(c) Claim: For every \( n \in \mathbb{N} \), if \( n \geq 4 \), then \( 2^n < n! \).
   Proof: The proof will be by induction on \( n \).
   Base case: \( 2^4 = 16 \) and \( 4! = 24 \) and \( 16 < 24 \), so the statement is true for \( n = 4 \).
   Induction step: Suppose \( k \in \mathbb{N} \) and \( 2^k < k! \). Then
   \[
   2^{k+1} = 2 \times 2^k < 2 \times k! < (k + 1) \times k! = (k + 1)!,
   
   so \( 2^{k+1} < (k + 1)! \). By the principle of mathematical induction, the statement is true for all \( n \geq 4 \).

   Answer: This proof receives an A. The reasoning is valid.

It is also fine to give an F because (1) \( k \geq 4 \) is not mentioned in the proof, or (2) there is no inductive hypothesis.

The proof fails to explicitly label the induction hypothesis. In this case, the inductive hypothesis is the assumption that \( 2^k < k! \), so the inductive hypothesis is present in the proof and the logic of the proof is valid. However, as a stylistic matter, it would have been better to label the inductive hypothesis explicitly, following the template described in class. This improvement might have made the reasoning in the proof a bit clearer.

(d) Claim: For all \( x, y, n \in \mathbb{N} \), if \( \max(x, y) = n \), then \( x \leq y \).
   Proof: The proof will be by induction on \( n \).
   Base case: Suppose that \( n = 0 \). If \( \max(x, y) = 0 \) and \( x, y \in \mathbb{N} \), then \( x = 0 \) and \( y = 0 \), hence \( x \leq y \).
   Inductive hypothesis: Assume that, whenever we have \( \max(x, y) = k \), then \( x \leq y \) must follow.
   Inductive step: We must prove that if \( \max(x, y) = k + 1 \), then \( x \leq y \). Suppose \( x, y \) are such that
max(x,y) = k + 1. Then it follows that max(x−1,y−1) = k, so by the inductive hypothesis, x−1 ≤ y−1. In this case, we have x ≤ y, completing the induction step. □

Answer: This proof receives an F. The problem lies in the application of the inductive hypothesis. More specifically, the incorrect step is: “Then it follows that max(x−1,y−1) = k−1, so by the inductive hypothesis, x−1 ≤ y−1.” The problem is that x−1 or y−1 might be negative (this happens when x = 0 or y = 0). Then the inductive hypothesis no longer applies, since x−1 and y−1 are not both natural numbers, so we cannot conclude that x−1 ≤ y−1.