Circuit N

Digraph G

Reduced Incidence Matrix A

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{bmatrix}
\]
Number of nodes: \( n = 3 \)
Number of branches: \( b = 4 \)
Number of circuit variables: \( 2b+(n-1) = (2\times4)+(3-1) = 10 \)
Number of Independent KCL Equations: \( n-1 = 2 \)
Number of Independent KVL Equations: \( b = 4 \)
Total number of independent KCL and KVL Equations: \( b+(n-1) = 6 \)
We need “\( b \)” \textbf{additional} independent equations in order to obtain a system of \( 2b+(n-1) \) \textbf{independent equations} in \( 2b+(n-1) \) circuit variables.

The additional equations must come from the \textbf{constitutive relation} which relate the terminal voltages and currents of the circuit elements.
KCL:

\[
\begin{bmatrix}
1 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\Rightarrow
i_1 + i_2 - i_4 = 0
- i_1 + i_3 = 0
\]

KVL:

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}
= \begin{bmatrix}
1 & -1 \\
1 & 0 \\
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\Rightarrow
v_1 = e_1 - e_2 \\
v_2 = e_1 \\
v_3 = e_2 \\
v_4 = -e_1
\]
Element Constitutive Relations

Element 1: Resistor
Described by Ohm’s Law: \( v_1 = 4 \, i_1 \)

Element 2: Resistor
Described by Ohm’s Law: \( v_2 = 3 \, i_2 \)

Element 3: Voltage source
Described by: \( v_3 = 5 \)

Element 4: Current source
Described by: \( i_4 = 2 \)

Rearranging these equations so that circuit variables appear on the left-hand side, we obtain

\[
\begin{align*}
\mathbf{v}_1 - 4 \, i_1 &= 0 \\
\mathbf{v}_2 - 3 \, i_2 &= 0 \\
\mathbf{v}_3 &= 5 \\
i_4 &= 2
\end{align*}
\]

Observe we have obtained 4 additional independent equations.

Equations obtained from the element constitutive relations are guaranteed to be independent because different elements involved different circuit variables.
Let us rearrange all 10 independent equations as follow:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
v_1 \\
v_2 \\
v_3 \\
v_4 \\
i_1 \\
i_2 \\
i_3 \\
i_4
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
2 \end{bmatrix}
\]
Let us rearrange all 10 independent equations as follow:

\[
\begin{bmatrix}
0 & 0 & A \\
-A^T & 1 & 0 \\
0 & H_v & H_i \\
\end{bmatrix}
= \begin{bmatrix}
e_1 \\
e_2 \\
v_1 \\
v_2 \\
v_3 \\
v_4 \\
i_1 \\
i_2 \\
i_3 \\
i_4 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
6 \\
2 \\
\end{bmatrix}
\]
The $(2b+(n-1)) \times (2b+(n-1))$ matrix $T$ is called the **tableau matrix** associated with the linear resistive circuit $\mathcal{N}$. 

The tableau equation is:

$$
\begin{bmatrix}
0 & 0 & A \\
-A^T & 1 & 0 \\
0 & H_v & H_i
\end{bmatrix}
\begin{bmatrix}
e \\
v \\
i
\end{bmatrix}
=
\begin{bmatrix}
0 \\
ü_v \\
ü_i
\end{bmatrix}
$$
KCL \[
\begin{align*}
    i_1 + i_2 - i_4 &= 0 \\
    -i_1 + i_3 &= 0
\end{align*}
\] (1) (2)

KVL \[
\begin{align*}
    v_1 &= e_1 - e_2 \\
    v_2 &= e_1 \\
    v_3 &= e_2 \\
    v_4 &= -e_1
\end{align*}
\] (3) (4) (5) (6)

Element \begin{align*}
    v_1 &= 4i_1 \\
    v_2 &= 3i_2 \\
    v_3 &= 6 \\
    i_4 &= 2
\end{align*}
\] (7) (8) (9) (10)

We can always find the solution using Cramer’s rule.