Solution: Problem Set 3

EECS123: Digital Signal Processing

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1. (a) We have,

\[(2M + 1)\sigma^2_x = \sum_{n=-M}^{M} x^2[n] - 2 \sum_{n=-M}^{M} \bar{x}x[n] + (2M + 1)\bar{x}^2,\]

\[= (2M + 1)P_x - 2\bar{x} \sum_{n=-M}^{M} x[n] + (2M + 1)\bar{x}^2,\]

\[= (2M + 1)P_x - 2(2M + 1)\bar{x}^2 + (2M + 1)\bar{x}^2,\]

\[= (2M + 1)(P_x - \bar{x}^2).\]

The result follows by cancelling \((2M + 1)\) from both sides.

(b) Following steps as in part (a), we can write,

\[E(x[n], c) = (P_x - 2c\bar{x} + c^2),\]

\[= (\bar{x} - c)^2 + P_x - \bar{x}^2.\]

The above quadratic is minimized at \(c = \bar{x}\), which is the desired answer.

Note: If you used the differentiation method, make sure you showed \(\frac{d^2}{dc^2}E(x[n], c) > 0\). Do not deduct points if you missed it in this homework, but make sure the mistake is not repeated in the future (for this course).

(c) Note that \((2M + 1)P_x = \frac{1}{2\pi} \int_{\pi}^{-\pi} |X(e^{j\omega})|^2d\omega\). And, \((2M + 1)\bar{x} = X(e^{j0}).\) Therefore, by using the formula derived in part (a)

\[\sigma^2_x = \frac{1}{(2M + 1)} \frac{1}{2\pi} \int_{\pi}^{-\pi} |X(e^{j\omega})|^2d\omega - \frac{1}{(2M + 1)^2} |X(e^{j0})|^2.\]

Note: If you have \((\bar{x})^2\) in the above expression, give yourself full point. Ideally you should have replaced \(\bar{x}\) by \(X(e^{j0})/(2M + 1)\).

2. (a) From property (i), \(h[n] = 0, \quad n < 0.\)

From property (ii), by conjugate symmetry of \(H(e^{j\omega})\), we know that \(h[n]\) is real.

From property (iii), since the DTFT of \(h[n + 1]\) is real, therefore, \(h[n + 1]\) sequence is even. Since \(h[n + 1] = 0, \quad n < -1,\) therefore, \(h[n] = 0, \quad n > 2.\) (Make a picture to convince yourself).

Thus, \(h[n]\) has length 3 and hence it is non-zero for a finite duration.
(b) Since \( h[n + 1] \) is even, therefore, \( h[0] = h[2] = a \). Let \( h[1] = b \). Then, using property (iv) and Parseval’s theorem, \( 2a^2 + b^2 = 2 \).

From property (v), \( a - b + a = 0 \), or \( 2a = b \).

Solving these equations, \( a = 1/\sqrt{3}, \ b = 2/\sqrt{3} \), OR \( a = -1/\sqrt{3}, \ b = -2/\sqrt{3} \). Thus, possible solutions to \( h[n] \) are,

\[
h[n] = \pm \left( \frac{1}{\sqrt{3}} \delta[n] + \frac{2}{\sqrt{3}} \delta[n - 1] + \frac{1}{\sqrt{3}} \delta[n - 2] \right).
\]

Note: Do not deduct half point if you gave only one solution out of the two (but be careful about it in the future). Also, the \( \pm \) operator should not be used on the individual co-efficients \( h[0], h[1], h[2] \). Either all of them are positive or all of them are negative. Individually, they cannot be positive or negative.

3. From Example 3.16 of Oppenheim and Schafer (pg. 122),

\[
X(z) = \log(1 + az^{-1}) \leftrightarrow (-1)^{n+1}a^n n u[n-1], \quad |z| > |a|.
\]

Thus,

\[
\log(1 + az^{-1}) - \log(1 + bz^{-1}) \leftrightarrow (-1)^{n+1}a^n n u[n-1] - (-1)^{n+1}b^n n u[n-1], \quad |z| > |b|.
\]

Finally, the left hand side can be simplified to,

\[
\log(1 + az^{-1}) - \log(1 + bz^{-1}) = \log \left( \frac{1 + az^{-1}}{1 + bz^{-1}} \right),
\]

\[
= \log \left( \frac{z + a}{z + b} \right).
\]

Thus,

\[
\log \left( \frac{z + a}{z + b} \right) \leftrightarrow (-1)^{n+1}a^n n u[n-1] - (-1)^{n+1}b^n n u[n-1], \quad |z| > |b|.
\]

With some care, power series method can be used to derive this relationship as well.

Note: If you expanded \( \log(z + a) \) or \( \log \left( 1 + \frac{z}{a} \right) \) in a power series, then it is not correct. \( |z| > |b| > |a| \) means that \( |z/a| > 1 \) and hence \( \log \left( 1 + \frac{z}{a} \right) \) will not have a power series expansion. The power series expansion for \( \log_\text{g}(1 + x) = x - x^2/2 + x^3/3 \ldots \) is valid only for \( |x| < 1 \).

Note: Give yourself 1/2 points if you got the + or − signs incorrect.

4. (a)

\[
Y(z) = X_1(z) + X_2(z) = \frac{z + z^{-2}}{z + 1} = \frac{(z^3 + 1)z^{-2}}{z + 1} = z^{-2}(z^3 - z + 1) = 1 - z^{-1} + z^{-2} \implies \text{ROC} = \{ z : z \neq 0 \}
\]
(b) \[ Y(z) = X(z)H(z) = \frac{1 + 2z^{-1}}{z^2 + 7z + 10} = \frac{z^{-1}(z + 2)}{z^2 + 7z + 10} = \frac{z^{-1}}{z + 5} \]

There is cancellation of poles, but not the critical pole at \( z = -5 \), which determines the ROC. Thus, \( ROC = \{ z : |z| > 5 \} \).

(c) \( Y(z) = z^{-2}X(z) \), i.e., \( y[n] = x[n - 2] \). Therefore, \( y[n] \) is \( x[n] \) shifted to the right by 2. Observe that \( x[n] \) is anti-causal with \( ROC_x = z : |z| < 2 \).

By shifting \( x[n] \) to the right, a non-negative term \( y[1] = x[-1] = 1/2 \) is induced. This introduces \( 1/2 \cdot z^{-1} \) term in \( Y(z) \). Therefore, \( ROC_y = \{ z : |z| < 2, z \neq 0 \} \).

5. We have,

\[
y_n = \frac{1}{4} \sum_{m=0}^{3} Y_m e^{j2\pi mn/4} \\
= \frac{1}{4} \sum_{m=0}^{3} X_{2m} e^{j2\pi mn/4} \\
= \frac{1}{4} \sum_{m=0}^{3} \sum_{l=0}^{7} x_l e^{-j2\pi(2m)l/8} e^{j2\pi mn/4} \\
= \frac{1}{4} \sum_{l=0}^{7} x_l \sum_{m=0}^{3} e^{j2\pi(m-n)/4} \\
= \frac{1}{4} \sum_{l=0}^{7} x_l \cdot 4(\delta(l - n) + \delta(l - n + 4)) \\
= x_n + x_{n+4}
\]

The next-to-last equality is tricky. Notice that the inner sum equals 4 if \( n - l = 0 \) or any multiple of 4 (it is zero otherwise). Because \( l \) is not restricted to the same range as \( m \), we end up with more than one delta function.

Final answer: \( b_k = a_k + a_{k+4} \) for \( k = 0, 1, 2, 3 \). This is an example of what is known as “time domain aliasing,” a dual concept to frequency domain aliasing.

6. (a) \{5, 7, 9, 5, 7, 9\}
(b) \{1, 2, 3, 5, 7, 9, 4, 5, 6\} (same as plain linear convolution)