
**Cayley-Hamilton Theorem.** Every square matrix $A$ satisfies its own characteristic equation:

$$\Delta(A) = 0$$

where the characteristic equation (aka characteristic polynomial) is given by:

$$\Delta(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + ... + c_1\lambda + c_0 = 0.$$  

*Proof* (For case when $A$ is similar to a diagonal matrix, i.e. for $A \in \mathbb{R}^{n \times n}$ with $A = P\Lambda P^{-1}$ where $\Lambda$ is a diagonal matrix with elements on the diagonal $\lambda_1, \lambda_2, ..., \lambda_n$.)

Substituting $A$ in the characteristic polynomial, we have

$$\Delta(A) = A^n + c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + ... + c_1A + c_0I$$

(1)

Noting that $A^k = PA^kP^{-1}$, then

$$\Delta(A) = P[A^n + c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + ... + c_1A + c_0I]P^{-1}.$$  

(2)

Since $\Lambda$ is diagonal, the typical $i,i$ term is given by

$$\Delta(\lambda_i) = |\lambda_i I - A| = \lambda_i^n + c_{n-1}\lambda_i^{n-1} + c_{n-2}\lambda_i^{n-2} + ... + c_1\lambda_i + c_0 = 0.$$  

Where the sum is zero because $\lambda_i$ is a root of the characteristic polynomial. Thus $\Delta(A) = P[0]P^{-1} = [0]$. □.

**Matrix Exponential** Recall series form for $e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + ...$ But from Cayley-Hamilton, we know that since $\Delta(A) = 0$ then $-A^n = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + ... + c_1A + c_0I$. And then all higher powers than $A^n$ can be expressed in terms of a linear sum of $I, A, A^2, ..., A^{n-1}$.

Then

$$e^{At} = a_0I + a_1At + ... + a_{n-1}(At)^{n-1} = R(A)$$

where $R(A)$ is a polynomial of degree $n-1$, and $a_i$ are found by solving $e^{\lambda_it} = a_0(t) + a_1(t)\lambda_i + ... + a_n(t)\lambda_i^n$.

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**Controllability**  
Assume that $\dot{x} = Ax + Bu$ is completely controllable. Recall that

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) d\tau.$$  

(3)

Since by assumption the system is controllable, we can choose a final time $t_1$ such that $x(t_1) = 0$ with initial condition $x(0) = x_0$ with $t_0 = 0$. So by eqn (3) we have

$$-x_0 = \int_{0}^{t_1} e^{-A\tau}Bu(\tau) d\tau.$$  

(4)

By Cayley-Hamilton, we can express $e^{-A\tau}$ as a polynomial in $A$:

$$e^{-A\tau} = a_0(\tau)I + a_1(\tau)A + a_2(\tau)A^2 + ... + a_{n-1}(\tau)A^{n-1} = \sum_{j=0}^{n-1} A^j a_j(\tau).$$  

(5)

If we substitute eqn (5) into eqn (4) we obtain

$$-x_0 = \sum_{j=0}^{n-1} A^j B \int_{0}^{t_1} a_j(\tau)u(\tau) d\tau.$$  

(6)

Note that $\int_{0}^{t_1} a_j(\tau)u(\tau) d\tau$ is a constant. Define $v_j = \int_{0}^{t_1} a_j(\tau)u(\tau) d\tau$. Then eqn (6) can be expressed as a matrix multiply:

$$-x_0 = [B]_{1 \times n}[AB]_{n \times n}B[...][A^{n-1}B].$$  

(7)

Define the controllability matrix $C = [B]_{1 \times n}[AB]_{n \times n}B[...][A^{n-1}B]$. Note that if state space is of dimension $n$, then eqn (7) will only be satisfiable for all $x_0$ if rank $(C) = n$. Thus the necessary condition for controllability is shown. □