1 Control Design — 25 points

Consider open loop plant

\[ G(s) = \frac{K(s^2 - 2s + 2)}{(s+2)(s+4)(s+5)(s+6)} \]

with unity feedback.

a) Sketch the root locus by hand, and verify using Matlab.

b) Find the range of gain, \( K \) that makes the system stable.

The closed loop transfer function is \( \Delta(s) = (s+2)(s+4)(s+5)(s+6) + K(s^2 - 2s + 2) \). We can find
the real-axis crossing by solving $\Delta(j\omega) = 0$ for real $\omega$ and $K$:

$$\Delta(j\omega) = (j\omega + 2)(j\omega + 4)(j\omega + 5)(j\omega + 6) + K((j\omega)^2 - 2j\omega + 2)$$

$$= 2K + 268j\omega - K\omega^2 - 17j\omega^3 - 104\omega^2 + \omega^4 - 2Kj\omega + 240$$

Real$(\Delta(j\omega)) = 2K - K\omega^2 - 104\omega^2 + \omega^4 + 240 = 0$

Imag$(\Delta(j\omega)) = 268\omega - 2K\omega - 17\omega^3 = 0$

Enforcing $K > 0$ results in two solutions: $(K = 115.58, \omega = 1.472)$ and $(K = 115.58, \omega = -1.472)$. Since we know the root locus is attracted to zeros, we can reason that $K < 115.58$ are stable and $K \geq 115.58$ causes the poles to cross into the RHP.

c) Using a second order approximation, find the value of $K$ that yields a closed-loop step response with 30% overshoot.

For 30% overshoot, we should search along a line of:

$$\zeta = \frac{-\ln(0.3)}{\sqrt{\pi^2 + \ln^2(0.3)}} \approx 0.3579$$

$$\theta \approx 69^\circ$$

Using MATLAB’s rlocus command,

```
hold on;
rlocus(zpk([1+1i,1-1i],[-2,-4,-5,-6],1),logspace(-2,10,100));
plot([0 10]*-cos(69*pi/180),[0 10]*sin(69*pi/180),'g');
ylim([-5 5]);
xlim([-10 5]);
```

we find that $k = 43.3$ lies roughly on this line.

d) Find all closed loop pole locations for $K$ found in part c)

From the rlocus plot, the dominant pole locations for this gain are at $-0.553 \pm 1.5j$.

e) Compare Matlab step response for $K$ found in part c) with second order approximation. Is the approximation used appropriate and accurate?

```
sys1 = zpk([],[-0.553+1.5j,-0.553-1.5j],1);
sys2 = feedback(zpk([1+1i 1-1i],[-2 -4 -5 -6],43.3),1);
step(sys1,syss2,[0:.01:10]);
legend('2nd order approx','Actual system');
```
For the transient response, the approximation fairly accurate although the actual system a) Lags a little because of the extra poles and b) Has a small “jerk” at the beginning due to the zeros.

2 Lead compensation — 25 points

Consider open loop plant

\[ G(s) = \frac{1}{(s + 3)(s + 5)} \]

Design goals: i) Settling time of 0.67 sec, and ii) per cent overshoot of 1.5%.

a) Show that the original system without compensation can not meet the transient specification.

The closed-loop response will be:

\[ G^{(closed)}(s) = \frac{K}{s^2 + 8s + 15 + K} = \frac{K}{15 + K} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

where

\[ \omega_n = \sqrt{15 + K} \]
\[ \zeta = \frac{4}{\sqrt{15 + K}} \]

To meet the design goals,

\[ T_s = \frac{4}{\zeta \omega_n} < 0.67 \]
\[ \%OS = 100 \exp -\left(\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}\right) < 1.5 \]

However,

\[ T_s = \frac{4}{\left(\sqrt{15 + K} \left(\frac{4}{\sqrt{15 + K}}\right)\right)} = 1 \]
Therefore, there is no $K$ such that the settling time will be met.

b) Show that a lead compensator $D(s) = K \frac{s + z}{s + p}$ with $z < p$ will meet the design specifications and find an acceptable set of values of $k$, $p$, and $z$. Verify with Matlab.

For a percent overshoot of 1.5%,

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.8$$
$$\theta = \cos^{-1}(0.8) = 36.8^\circ$$

So, we “slide” down this line until we reach a settling time of $T_s = 0.67$. This gives the point, along with the $36.8^\circ$ line, which defines the edge of the acceptable region for poles of the second-order approximation.

$$\text{Real}(s) = -\zeta \omega_n = -\frac{4}{T_s} = -5.97$$
$$\text{Imag}(s) = 5.97 \tan(36.8^\circ) = 4.4664$$

Now comes the question of choosing where to place the zero and pole and the proportional gain of the system, $k$. There are many ways to go about this, described is one way:

Choose the zero to be at -10 to attract the root locus towards it. The placement of the pole will determine the rate of which the zero is “cancelled” as $k$ increases. We can choose it to be -40 to give the zero ample time to act to bring the root locus towards the left.

Now, we choose a $k$ for which the root locus will cross into the desired region. The overshoot will be O.K. until the other branches threaten to cross the $36.8^\circ$ line. It seems that values of $k$ between about 20 and 35 can work for this setup. We can find this by the rlocus command in matlab. A value of $k = 572$ gives a damping of 0.805, overshoot of 1.41% and a settling time of 0.25s.

To summarize:

$$z = 10$$
$$p = 40$$
$$k = 572$$

C) Hand sketch the root locus for the original system and the system with a lead compensator, and verify with Matlab.
d) What is the steady state error $e(t)$ for the uncompensated and compensated systems?

With a gain of 572, steady state error for the compensated system is 9.5% and steady state error for uncompensated system is 2.7%.

3 Bode Plot — 30 points

Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. Verify sketch using MATLAB plot with same axes scales, and turn in.

a) $\frac{s^2 + 2s + 101}{s^3 + 101}$
\[ G(s) = \frac{s^2 + 2s + 101}{s^3} = \left( \frac{s^2}{101} + 2 \frac{1}{\sqrt{101}} \frac{s}{\sqrt{101}} + 1 \right) \cdot 101 \cdot \frac{1}{s^3} \]  

(1)

Zeros: second-order zeros with \( \zeta = \frac{1}{\sqrt{101}} \), \( \omega_n = \sqrt{101} \approx 10 \)

Poles: three first-order poles at \( s = 0 \)

Start the graph off at \( \omega = 1 \), a decade below the breakpoint of the zeros. Evaluate \( G(j1) \). We can ignore the second-order zeros here; the gain is 101, or approx. 40 dB. The phase is -270° (-90° for each of the poles).

The magnitude asymptote is originally sloped at -60 dB/dec (three poles), until the breakpoint of the zeros, after which the slope is -20 dB/dec. The phase changes from -270° to -90° over two decades \( (s = j1 \text{ to } s = j100) \).

\[ G(s) = \frac{10^3}{(s+1)(s^2 + 2s + 101)} = \frac{1000}{101} \frac{1}{s+1} \frac{1}{s^2} + 2 \frac{1}{\sqrt{101}} \frac{s}{\sqrt{101}} + 1 \]  

(2)

Zeros: none

Poles: a second-order pair with \( \zeta = \frac{1}{\sqrt{101}} \), \( \omega_n = \sqrt{101} \approx 10 \); a first-order pole at \( s = -1 \).

Start the plot at low frequencies \( (j\omega \text{ approaches zero}) \). \( G(j\omega \rightarrow 0) \approx \frac{1000}{101} \approx 20 \text{ dB} \). The phase at low frequencies is zero. The first-order pole has its breakpoint at \( \omega = 1 \) while the second-order pole pair have their breakpoint at \( \omega \approx 10 \).
c) $\frac{s+1}{(s+10)(s+30)}$

$$G(s) = \frac{s+1}{(s+10)(s+30)} = \frac{1}{300} \left( \frac{s+1}{s+10} + \frac{1}{s+30} \right)$$

Zeros: one first-order zero at $s = -1$
Poles: one first-order pole at $s = -10$ and one first-order pole at $s = -30$.
At low frequencies, $G(j\omega) \approx \frac{1}{300} \approx -50\text{ dB}$; the phase is zero.

d) $\frac{s^2+40s+10^4}{s^2+10s+100}$
\[ G(s) = \frac{s^2 + 40s + 10^4}{s^2 + 10s + 100} = \frac{10^4}{100} \left( \frac{s^2 + 2 \cdot \frac{40}{10^2} s + 1}{s + 2 \cdot \frac{5}{10} 10s + 100} \right) \]

(4)

Zeros: second order pair with \( \zeta = \frac{40}{100}, \omega_n = 100 \)
Poles: second order pair with \( \zeta = \frac{5}{10}, \omega_n = 10 \)
At low frequencies, \( G(j\omega) \approx 100 \approx 40 \text{ dB}, \) zero phase.

e) \[ \frac{10^4}{(s + 0.1)(s + 3)(s + 30)} \]

\[ G(s) = \frac{10^4}{(s + 0.1)(s + 3)(s + 30)} = \frac{10^4}{0.1 \cdot 3 \cdot 30} \left( \frac{1}{s + 0.1} + \frac{1}{s + 3} + \frac{1}{s + 30} + 1 \right) \]

(5)

Zeros: none
Poles: first order poles at \( s = -0.1, -3, -30 \)
At low frequencies, \( G(j\omega) \approx 1111 \approx 60 \text{ dB}, \) zero phase.
4 Compensation Network — 20 points

For the ideal op amp circuit:

a) Determine the transfer function $T(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}$.

Use KCL at the negative terminal of the op amp.

\[
\frac{V_{\text{in}}(s)}{R_2 + \frac{1}{C_2 s}} + \frac{V_{\text{out}}(s)}{R_1 + \frac{1}{R_3 + C_1 s}} = 0
\]

\[
\frac{-V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_1 + \frac{1}{R_3 + C_1 s}}{R_2 + \frac{1}{C_2 s}}
\]

\[
\frac{-V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_1 C_2 s \left( \frac{1}{R_3} + C_1 s \right) + C_2 s}{(R_2 C_2 s + 1) \left( \frac{1}{R_3} + C_1 s \right)}
\]

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{-R_1 C_1 C_2 s^2 - \left( \frac{R_1 C_2}{R_3} + C_2 \right) s}{R_2 C_1 C_2 s^2 + \left( \frac{R_1 C_2}{R_3} + C_1 \right) s + \frac{1}{R_3}}
\]

b) Hand sketch the Bode plot for magnitude and phase for $R_1 = 1K$ $\Omega$, $R_2 = 10K$ $\Omega$, $R_3 = 100K$ $\Omega$, $C_1 = 1000$ nF, and $C_2 = 1000$ nF.

Replace the values in the transfer function above with the component values.

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{-\left( 10^3 \times 10^{-6} \times 10^{-6} \right) s^2 - \left( 10^3 \times 10^{-6} \times 10^{-5} + 10^{-6} \right) s}{\left( 10^4 \times 10^{-6} \times 10^{-6} \right) s^2 + \left( 10^4 \times 10^{-6} \times 10^{-5} + 10^{-6} \right) s + (10^{-5})}
\]

\[
= \frac{-\left( 10^{-9} \right) s^2 - (1.01 \times 10^{-6}) s}{\left( 10^{-8} \right) s^2 + (1.1 \times 10^{-6}) s + (10^{-5})}
\]

9
Manipulate the TF to break it into standard forms

\[
= -s(10^{-9}s + 1.01 \times 10^{-6}) \cdot \frac{10^8}{s^2 + 110s + 1000}
\]

\[
= \frac{-(1010)(10^5)}{10^9} s \left( \frac{s}{1010} + 1 \right) \frac{1}{s^2 + 2 \frac{55}{\sqrt{1000}} \frac{s}{\sqrt{1000}} + 1}
\]

Zeros: first-order zeros at \( s = 0, s = -1010 \)

Poles: second-order pair with \( \zeta = \frac{55}{\sqrt{1000}} \approx 1.74, \omega_n = \sqrt{1000} \approx 31.6. \)

Because of the zero at \( s = 0 \), we can’t start the plot at “low frequencies”. Instead we must choose a \( \omega \) small enough so that the other poles/zeros can be ignored, and evaluate there. \( \omega = 1 \) is more than a decade lower than everything else. \( G(j1) \approx -0.101 \): so magnitude -20dB, phase 270° (180° for the negative sign, 90° for the first-order zero). The slope of the magnitude in this region is 20 dB/dec because of the zero.

c) Verify sketch using MATLAB plot with same axes scales, and turn in.