Poisson Distribution

Poisson distribution:
a given event occurring in any time interval is distributed
uniformly over the interval.

The probability of \( n \) electrons arriving in a period \( T \): is

\[
p(n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}
\]

where \( \bar{n} \) is the average number of electrons arriving in \( T \)

Properties of Poisson Distribution:
Mean = \( \bar{n} \)
Variance = \( \bar{n} \)
Spectral Density Function

Random variable \( i(t) \) consists of a large number of individual events (e.g., single-electron photocurrent) at random time:

\[
i(t) = \sum_{j=1}^{N_t} f(t - t_j), \quad 0 \leq t \leq T
\]

Fourier transform: \( I_r(\omega) = \sum_{j=1}^{N_t} F_r(\omega) \)

\[
F_r(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t - t_j)e^{-i\omega t}dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = e^{-i\omega t_j} F(\omega)
\]

\[
\sum_{j=1}^{N_t} \sum_{j=1}^{N_t} e^{-i\omega(t_j - t_j')} = N_t |F(\omega)|^2 = NT|F(\omega)|^2
\]

\( N_t \): average rate of electron arrival

Spectral density function: \( S(\nu) = \lim_{T \to \infty} \frac{1}{2T} |I_r(2\pi \nu)|^2 = \frac{8\pi^2 N_t}{T} |F(2\pi \nu)|^2 \)

Shot Noise

Shot Noise: Noise current arising from random generation and flow of mobile charge carriers.

Current pulse due to a single electron moving at \( \nu(t) \):

\[
i_e(t) = \frac{e\nu(t)}{d}
\]

Fourier transform: \( F(\omega) = \frac{1}{2\pi} \int_0^\infty \nu(t)e^{-i\omega t}dt \)

\( t_a \): arrival time, \( x(0) = 0 \), \( x(t_a) = d \),

Small transit time \( t_a \), \( \omega t_a << 1 \) \( \rightarrow e^{-i\omega t} \approx 1 \)

\[
F(\omega) = \frac{1}{2\pi} \int_0^\omega dx \cdot \epsilon dx = \frac{1}{2\pi} \int_0^\omega dx = \frac{\epsilon}{2\pi}
\]

\[
S(\nu) = 8\pi^2 N \left( \frac{\epsilon}{2\pi} \right)^2 = 2\epsilon \bar{I}
\]

\[\bar{I} = eN\]

\[
\frac{S(\nu)}{2e} = S(\nu)dv = 2\epsilon \bar{I}dv
\]
Thermal Noise (Johnson Noise)

- Fluctuation in the voltage across a dissipative circuit element (resistor)
- Caused by thermal motion of charged carriers

Thermal Noise Derivation

Consider two resistors connected by a lossless transmission line of length $L$:

- Voltage wave: $v(t) = A \cos(\omega t \pm kz)$

Assume periodic condition: $kL = 2m\pi$

Mode density: $\rho(\nu) = \frac{L}{c}$

Power flow: $P = \frac{\text{Energy}}{\text{Transit Time}}$

$$P = \frac{1}{L/c} \left( \frac{L}{c} \right) \left( \frac{hv}{e^{hv/k_BT} - 1} \right) = \frac{hv\Delta \nu}{e^{hv/k_BT} - 1}$$

$hv / k_BT \ll 1$

$$P = k_BT \Delta \nu = \left( \frac{1}{2} \left( \frac{R}{R+R} \right) \right) \left( \frac{1}{2} \left( \frac{R}{R+R} \right) \right)^2 R$$

Equivalent mean square noise voltage:

$$\overline{\nu^2} = 4k_BT \Delta \nu$$

Equivalent mean square noise current:

$$\overline{i^2} = \frac{4k_BT \Delta \nu}{R}$$
Noise in p-i-n Photodiode

Noises in p-i-n photodiodes: shot noise and thermal noise

\[ \tilde{i}_N(v) = \tilde{i}_{N,\text{shot}}(v) + \tilde{i}_{N,\text{thermal}}(v) = 2e\tilde{I}dv + \frac{4k_B T \Delta v}{R} \]

Signal:

\[ \tilde{i}_S(v) = \bar{I} \]

Signal to noise ratio (SNR):

\[ \text{SNR} = \frac{\bar{I}^2}{2e\tilde{I}dv + \frac{4k_B T \Delta v}{R}} \]

Note that the SNR improves with increasing average photocurrent \( \bar{I} \)

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**Example**

\( B = 1 \text{ GHz} \)
\( R = 50 \Omega \)

![Example Graphs](image)