EE27 - Lecture 2
Filters

- From last lecture:
  - Dynamic range of analog circuits

- Filters:
  - Nomenclature
  - Specifications
    - Quality factor
    - Frequency characteristics
    - Group delay
  - Filter types
    - Butterworth
    - Chebyshev I & II
    - Elliptic
    - Bessel
  - Group delay comparison example
  - Biquads

Nomenclature
Filter Types

- Lowpass
- Highpass
- Bandpass
- Band-reject (Notch)
- All-pass

Provide frequency selectivity
Phase shaping or equalization
Filter Specifications

- Frequency characteristics (lowpass filter):
  - Passband ripple (Rpass)
  - Cutoff frequency or -3dB frequency
  - Stopband rejection
  - Passband gain
- Phase characteristics:
  - Group delay
- SNR (Dynamic range)
- SNDR (Signal to Noise+Distortion ratio)
- Linearity measures: IM3 (intermodulation distortion), HD3 (harmonic distortion), IIP3 or OIP3 (Input-referred or output-referred third order intercept point)
- Power/pole & Area/pole

Lowpass Filter Frequency Characteristics
Quality Factor ($Q$)

- The term quality factor ($Q$) has different definitions in different contexts:
  - Component quality factor (inductor & capacitor $Q$)
  - Pole quality factor
  - Bandpass filter quality factor

- Next 3 slides clarifies each

Component Quality Factor ($Q$)

- For any component with a transfer function:
  
  \[ H(j\omega) = \frac{1}{R(\omega) + jX(\omega)} \]

- Quality factor is defined as:
  
  \[ Q = \frac{X(\omega)}{R(\omega)} \rightarrow \frac{\text{Energy Stored}}{\text{Average Power Dissipation}} \text{ per unit time} \]
Inductor & Capacitor Quality Factor

- Inductor $Q_L$:
  \[ Y_L = \frac{1}{R_s + j\omega L} \quad Q_L = \frac{\omega L}{R_s} \]

- Capacitor $Q_C$:
  \[ Z_C = \frac{1}{R_p + j\omega C} \quad Q_C = \omega C R_p \]

Pole Quality Factor

\[ Q_{Pole} = \frac{\omega_p}{2\sigma_x} \]
Bandpass Filter Quality Factor ($Q$)

![Diagram of Bandpass Filter Quality Factor](image)

$Q = \frac{f_{\text{center}}}{\Delta f}$

What is Group Delay?

- Consider a continuous time filter with s-domain transfer function $G(s)$:

$$G(j\omega) = |G(j\omega)| e^{j\theta(\omega)}$$

- Let us apply a signal to the filter input composed of sum of two sinewaves at slightly different frequencies ($\Delta \omega << \omega$):

$$v_{\text{in}}(t) = A_1 \sin(\omega t) + A_2 \sin[(\omega + \Delta \omega) t]$$

- The filter output is:

$$v_{\text{out}}(t) = A_1 |G(j\omega)| \sin(\omega t + \theta(\omega)) + A_2 |G[j(\omega + \Delta \omega)]| \sin[(\omega + \Delta \omega)t + \theta(\omega + \Delta \omega)]$$
What is Group Delay?

\[ v_{\text{out}}(t) = A_1 |G(j\omega)| \sin \left\{ \omega \left[ t + \frac{\theta(\omega)}{\omega} \right] \right\} + \]

\[ + A_2 |G[j(\omega+\Delta\omega)]| \sin \left\{ (\omega+\Delta\omega) \left[ t + \frac{\theta(\omega+\Delta\omega)}{\omega+\Delta\omega} \right] \right\} \]

Since \( \frac{\Delta\omega}{\omega} \ll 1 \) then \( \left[ \frac{\Delta\omega}{\omega} \right]^2 \rightarrow 0 \)

\[ \frac{\theta(\omega+\Delta\omega)}{\omega+\Delta\omega} \equiv \left[ \theta(\omega) + \frac{d\theta(\omega)}{d\omega} \Delta\omega \right] \left[ \frac{1}{\omega} \left( 1 - \frac{\Delta\omega}{\omega} \right) \right] \]

\[ \equiv \frac{\theta(\omega)}{\omega} + \left( \frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega} \right) \frac{\Delta\omega}{\omega} \]

What is Group Delay?

Signal Magnitude and Phase Impairment

\[ v_{\text{out}}(t) = A_1 |G(j\omega)| \sin \left\{ \omega \left[ t + \frac{\theta(\omega)}{\omega} \right] \right\} + \]

\[ + A_2 |G[j(\omega+\Delta\omega)]| \sin \left\{ (\omega+\Delta\omega) \left[ t + \frac{\theta(\omega)}{\omega} + \left( \frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega} \right) \frac{\Delta\omega}{\omega} \right] \right\} \]

- If the second term in the phase of the 2nd sin wave is non-zero, then the filter’s output at frequency \( \omega+\Delta\omega \) is time-shifted differently than the filter’s output at frequency \( \omega \) → “Phase distortion”
- If the second term is zero, then the filter’s output at frequency \( \omega+\Delta\omega \) and the output at frequency \( \omega \) are each delayed in time by \( -\theta(\omega)/\omega \)
- \( \tau_{\text{PD}} = -\theta(\omega)/\omega \) is called the “phase delay” and has units of time
What is Group Delay?  
Signal Magnitude and Phase Impairment

- Phase distortion is avoided only if:
  \[
  \frac{d\theta(\omega)}{d\omega} \cdot \frac{\theta(\omega)}{\omega} = 0
  \]

- Clearly, if \( \theta(\omega) = k\omega \), \( k \) a constant, \( \rightarrow \) no phase distortion
- This type of filter phase response is called "linear phase" \( \rightarrow \)Phase shift varies linearly with frequency
- \( \tau_{GR} \equiv -\frac{d\theta(\omega)}{d\omega} \) is called the "group delay" and also has units of time. For a linear phase filter \( \tau_{GR} = \tau_{PD} = k \)
  \( \rightarrow \) \( \tau_{GR} = \tau_{PD} \) implies linear phase
- Note: Filters with \( \theta(\omega) = k\omega + c \) are also called linear phase filters, but they’re not free of phase distortion

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What is Group Delay?  
Signal Magnitude and Phase Impairment

- If \( \tau_{GR} = \tau_{PD} \rightarrow \) No phase distortion

\[
v_{out}(t) = A_1 |G(j\omega)| \sin \left[ \omega \left( t - \tau_{GR} \right) \right] +
\]
\[
+ A_2 |G[j(\omega+\Delta\omega)]| \sin \left[ (\omega+\Delta\omega) \left( t - \tau_{GR} \right) \right]
\]

- If also \( |G(j\omega)| = |G[j(\omega+\Delta\omega)]| \) for all input frequencies within the signal-band, \( v_{out} \) is a scaled, time-shifted replica of the input, with no "signal magnitude distortion"
- In most cases neither of these conditions are exactly realizable
Phase delay is defined as:
\[ \tau_{PD} \equiv -\frac{\theta(\omega)}{\omega} \quad \text{[time]} \]

Group delay is defined as:
\[ \tau_{GR} \equiv -\frac{d\theta(\omega)}{d\omega} \quad \text{[time]} \]

If \( \theta(\omega)=k\omega \), \( k \) a constant, \( \rightarrow \) no phase distortion

For a linear phase filter \( \tau_{GR} \equiv \tau_{PD} = k \)

Summary

Group Delay

Filter Types

Lowpass Butterworth Filter

- Maximally flat amplitude within the filter passband

\[ \frac{d^N |H(j\omega)|}{d\omega} \bigg|_{\omega=0} = 0 \]

- Moderate phase distortion

Example: 5th Order Butterworth filter
**Lowpass Butterworth Filter**

- All poles
- Poles located on the unit circle with equal angles

**Example: 5th Order Butterworth Filter**

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**Filter Types**

**Chebyshev I Lowpass Filter**

- Chebyshev I filter
  - Ripple in the passband
  - Sharper transition band compared to Butterworth
  - Poorer group delay
  - As more ripple is allowed in the passband:
    - Sharper transition band
    - Poorer phase response

**Example: 5th Order Chebyshev filter**
**Chebyshev I Lowpass Filter Characteristics**

- All poles
- Poles located on an ellipse inside the unit circle
- Allowing more ripple in the passband:
  - Narrower transition band
  - Sharper cut-off
  - Higher pole Q
  - Poorer phase response

**Example: 5th Order Chebyshev I Filter**

<table>
<thead>
<tr>
<th>Normalized Frequency</th>
<th>Phase (deg)</th>
<th>Magnitude (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>-90</td>
<td>-20</td>
</tr>
<tr>
<td>1</td>
<td>-180</td>
<td>-40</td>
</tr>
<tr>
<td>1.5</td>
<td>-270</td>
<td>-60</td>
</tr>
<tr>
<td>2</td>
<td>-360</td>
<td>-80</td>
</tr>
</tbody>
</table>

**Filter Types**

**Chebyshev II Lowpass**

- Chebyshev II filter
  - No ripple in passband
  - Nulls or notches in stopband
  - Sharper transition band compared to Butterworth
  - Passband phase more linear compared to Chebyshev I

**Example: 5th Order Chebyshev II filter**
Filter Types
Chebyshev II Lowpass

- Both poles & zeros
  - No. of poles $n$
  - No. of finite zeros $n-1$
- Poles located both inside & outside of the unit circle
- Complex conjugate zeros located on $j\omega$ axis
- Nulls in stopband

Filter Types
Elliptic Lowpass Filter

- Elliptic filter
  - Ripple in passband
  - Nulls in the stopband
  - Sharper transition band compared to Butterworth & both Chebyshevs
  - Poorest phase response
Filter Types
Elliptic Lowpass Filter

- Both poles & zeros
  - No. of poles: $n$
  - No. of finite zeros: $n-1$
- Zeros located on $j\omega$ axis
- Sharp cut-off
  - Narrower transition band
  - Pole Q higher compared to the previous filters

Example: 5th Order Elliptic Filter

Filter Types
Bessel Lowpass Filter

- Bessel
  - All poles
  - Maximally flat group delay
  - Poor out-of-band attenuation
  - Poles outside unit circle
  - Relatively low Q poles

Example: 5th Order Bessel filter
Magnitude Response of a Bessel Filter as a Function of Filter Order (n)

Filter Types
Comparison of Various Type LPF Magnitude Response

All 5th order filters with same corner freq.
Filter Types
Comparison of Various LPF Singularities

- Poles Bessel
- Poles Butterworth
- Poles Elliptic
- Zeros Elliptic
- Poles Chebyshev I 0.1dB

Comparison of Various LPF Groupdelay

Group Delay Comparison
Example

- Lowpass filter with 100kHz corner frequency
- Chebyshev I versus Bessel
  - Both filters 4th order - same -3dB point
  - Passband ripple of 1dB allowed for Chebyshev I

Magnitude Response
4th Order Chebyshev I versus Bessel
Phase Response
4th Order Chebyshev I versus Bessel

Group Delay
4th Order Chebyshev I versus Bessel
Normalized Group Delay
4th Order Chebyshev I versus Bessel

Step Response
4th Order Chebyshev I versus Bessel
**Intersymbol Interference (ISI)**

ISI → Broadening of pulses resulting in interference between successive transmitted pulses

Example: Simple RC filter

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**Pulse Impairment**

Bessel versus Chebyshev

Note that in the case of the Chebyshev filter not only the pulse has broadened but it also has a long tail

→ More ISI compared to Bessel
**Response to Pseudo-Random Data**

**Chebyshev versus Bessel**

Input Signal:
Symbol rate $1/130k$Hz

<table>
<thead>
<tr>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.5</td>
<td>-1</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
<td>1.5</td>
<td></td>
</tr>
</tbody>
</table>

**Summary**

**Filter Types**

- Filters with high signal attenuation per pole $\Rightarrow$ poor phase response
- For a given signal attenuation, requirement of preserving constant group delay $\Rightarrow$ Higher order filter
  - In the case of passive filters $\Rightarrow$ higher component count
  - For integrated active filters $\Rightarrow$ higher chip area & power dissipation
- In cases where filter is followed by ADC and DSP
  - Possible to digitally correct for phase impairments incurred by the analog circuitry by using digital phase equalizers
RLC Filters

• Bandpass filter:

\[
\frac{V_o}{V_{in}} = \frac{sRC}{s^2 + \omega_0^2 s + \omega_0^2}\]

\[
\omega_0 = \frac{1}{\sqrt{LC}}
\]

\[
Q = \omega_0 R C = \frac{R}{L \omega_0}
\]

Singularities: Pair of complex conjugate poles
Zeros @ \( f=0 \) & \( f=\infty \).

• Design a bandpass filter with:
  - Center frequency of 1kHz
  - Quality factor of 20

• First assume the inductor is ideal
• Next consider the case where the inductor has series \( R \) resulting in a finite inductor \( Q \) of 40
• What is the effect of finite inductor \( Q \) on the overall \( Q \)?
RLC Filters

Effect of Finite Component $Q$

$$\frac{1}{Q_{\text{filt}}} = \frac{1}{Q_{\text{filt}}^{\text{ideal}}} + \frac{1}{Q_{\text{Ind.}}}$$

![Graph showing the effect of finite component Q on filter response.]

$Q_{\text{filt}} = 20$ (ideal L)

$Q_{\text{filt}} = 13.3$ ($Q_L = 40$)

$\Rightarrow$ Need to have component $Q$ much higher compared to desired filter $Q$

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RLC Filters

Question:
Can RLC filters be integrated on-chip?
Monolithic Inductors
Feasible Quality Factor & Value

Feasible monolithic inductor in CMOS tech. <10nH with Q < 7

Ref: “Radio Frequency Filters”, Lawrence Larson; Mead workshop presentation 1999

Monolithic LC Filters

- Monolithic inductor in CMOS tech.
  - L < 10nH with Q < 7

- Max. capacitor size (based on realistic chip area)
  - C < 20pF

LC filters in the monolithic form feasible:
- Frequency > 350MHz
- Only low quality factor filters

Learn more in EE242
Integrated Filters

- Implementation of RLC filters in CMOS technologies requires on-chip inductors
  - Integrated L<10nH with Q<10
  - Combined with max. cap. 20pF
→ LC filters in the monolithic form feasible: freq>350MHz

- Analog/Digital interface circuitry require fully integrated filters with critical frequencies << 350MHz

- Hence:

  ⇒ Need to build active filters built without inductor

Filters

2nd Order Transfer Functions (Biquads)

- Biquadratic (2nd order) transfer function:

\[
H(s) = \frac{1}{1 + \frac{s}{\omega_p Q_p} + \frac{s^2}{\omega_p^2}}
\]

\[
|H(j\omega)| = \frac{1}{\sqrt{1 - \frac{\omega^2}{\omega_p^2} + \left(\frac{\omega}{\omega_p Q_p}\right)^2}}
\]

\[
Biquad poles @: s = -\frac{\omega_p}{2Q_p} \left(1 \pm i\sqrt{1 - 4Q_p^2}\right)
\]

Note: for \(Q_p \leq \frac{1}{2}\) poles are real, complex otherwise
Biquad Complex Poles

\[ Q_P > \frac{1}{2} \quad \rightarrow \quad \text{Complex conjugate poles:} \]

\[ s = -\frac{\omega_P}{2Q_P} \left( 1 \pm j\sqrt{4Q_P^2 - 1} \right) \]

Distance from origin in s-plane:

\[ d^2 = \frac{\left( \frac{\omega_P}{2Q_P} \right)^2}{1 + 4Q_P^2 - 1} = \omega_P^2 \]

\[ s = -\frac{\omega_P}{2Q_P} \left( 1 \pm j\sqrt{4Q_P^2 - 1} \right) \]
Implementation of Biquads

- Passive RC: only real poles → can’t implement complex conjugate poles
- Terminated LC
  - Low power, since it is passive
  - Only fundamental noise sources → load and source resistance
  - As previously analyzed, not feasible in the monolithic form for \( f < 250\text{MHz} \)
- Active Biquads
  - Many topologies can be found in filter textbooks!
  - Widely used topologies:
    - Single-opamp biquad: Sallen-Key
    - Multi-opamp biquad: Tow-Thomas
    - Integrator based biquads

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Active Biquad

Sallen-Key Low-Pass Filter

\[
H(s) = \frac{G}{1 + \frac{s}{\omega_p Q_p} + \frac{s^2}{\omega_p^2}}
\]

\[
\omega_p = \frac{1}{\sqrt{R_1 C_1 R_2 C_2}}
\]

\[
Q_p = \frac{\omega_p}{\frac{1}{R_0 C_1} + \frac{1}{R_2 C_2} + \frac{1-G}{R_0 C_2}}
\]

- Single gain element
- Can be implemented both in discrete & monolithic form
- “Parasitic sensitive”
- Versions for LPF, HPF, BP, ...
  - Advantage: Only one opamp used
  - Disadvantage: Sensitive to parasitic – all pole no zeros
Addition of Imaginary Axis Zeros

- Sharpen transition band
- Can "notch out" interference
- High-pass filter (HPF)
- Band-reject filter

\[ H(s) = K \left( 1 + \frac{s^2}{\omega_Z} \right)^2 \]

\[ H(j\omega) \big|_{\omega \rightarrow \infty} = K \left( \frac{\omega_p}{\omega_Z} \right)^2 \]

Note: Always represent transfer functions as a product of a gain term, poles, and zeros (pairs if complex). Then all coefficients have a physical meaning, and readily identifiable units.

Imaginary Zeros

- Zeros substantially sharpen transition band
- At the expense of reduced stop-band attenuation at high frequency

\[ f_p = 100kHz \]
\[ Q_z = 2 \]
\[ f_z = 3f_p \]
Moving the Zeros

\[ f_p = 100kHz \]
\[ Q_p = 2 \]
\[ f_z = f_p \]

Frequency Response

\[
\frac{V_{o1}}{V_{in}} = -k_2 \frac{(b_2a_1 - b_1)s + (b_2a_0 - b_0)}{s^2 + a_1s + a_0}
\]

\[
\frac{V_{o2}}{V_{in}} = \frac{b_2s^2 + b_1s + b_0}{s^2 + a_1s + a_0}
\]

\[
\frac{V_{o3}}{V_{in}} = -\frac{1}{k_1\sqrt{a_0}} \frac{(b_0 - b_2a_0)s + (a_1b_0 - a_0b_1)}{s^2 + a_1s + a_0}
\]

- \( V_{o2} \) implements a general biquad section with arbitrary poles and zeros
- \( V_{o1} \) and \( V_{o3} \) realize the same poles but are limited to at most one finite zero

Component Values

\[
b_0 = \frac{R_k}{R_R R_R C_1 C_2}
\]

\[
b_1 = \frac{1}{R_R C_1} \left( \frac{R_k}{R_k R_k R_R C_2} \right)
\]

\[
b_2 = \frac{R_k}{R_k R_k R_R C_1 C_2}
\]

\[
a_k = \frac{1}{R_R R_R C_1 C_2}
\]

\[
a_1 = \frac{1}{R_R C_1}
\]

\[
k_1 = \sqrt{\frac{R_R R_R C_1}{R_R R_R C_2}}
\]

\[
k_2 = \frac{R_k}{R_k R_k}
\]

given \( a_1, b_1, k_1, C_1, C_2 \) and \( R_k \)

\[
R_1 = \frac{1}{a_1 C_1}
\]

\[
R_2 = -\frac{k_1}{\sqrt{a_1 C_1}}
\]

\[
R_3 = \frac{1}{k_1 k_2 \sqrt{a_1 C_1}}
\]

\[
R_4 = \frac{1}{k_1 a_1 b_1 - b_1 C_1}
\]

\[
R_5 = \frac{k_1 \sqrt{a_1}}{b_1 C_1}
\]

\[
R_6 = \frac{R_k}{R_k}
\]

\[
R_s = k_2 R_k
\]

\[
R_s = k_2 R_k
\]

it follows that

\[
\omega_r = \frac{R_k}{\sqrt{R_s R_s R_R R_R C_1 C_2}}
\]

\[
Q_r = \omega_r R_k C_1
\]