Quick Warm-Up

- Suppose we have a ***biased*** coin that comes up heads with some ***unknown*** probability p; how can we use it to produce random bits with probabilities of ***exactly*** 0.5 for 0 and 1?

Quick Warm-Up

- Suppose we have a ***biased*** coin that comes up heads with some ***unknown*** probability p; how can we use it to produce random bits with probabilities of ***exactly*** 0.5 for 0 and 1?
- Answer (von Neumann):
  - Flip coin twice, repeat until the outcomes are different
  - HT = 0, TH = 1, each has probability $p(1-p)$

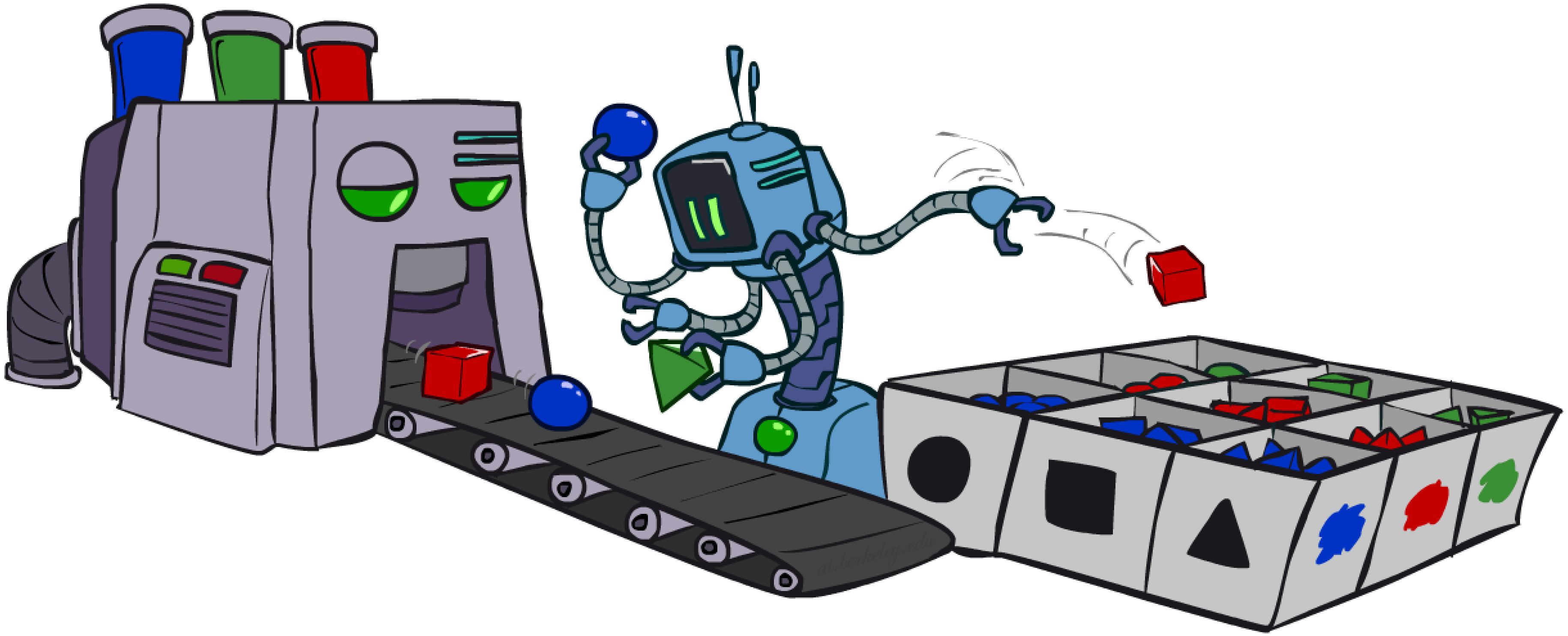
Bayes Nets

- ✓ Part I: Representation
- ✓ Part II: Exact inference
  - ✓ Enumeration (always exponential complexity)
  - ✓ Variable elimination (worst-case exponential complexity, often better)
  - ✓ Inference is NP-hard in general
- Part III: Approximate Inference

Later: Learning Bayes nets from data

CS 188: Artificial Intelligence

Bayes Nets: Approximate Inference



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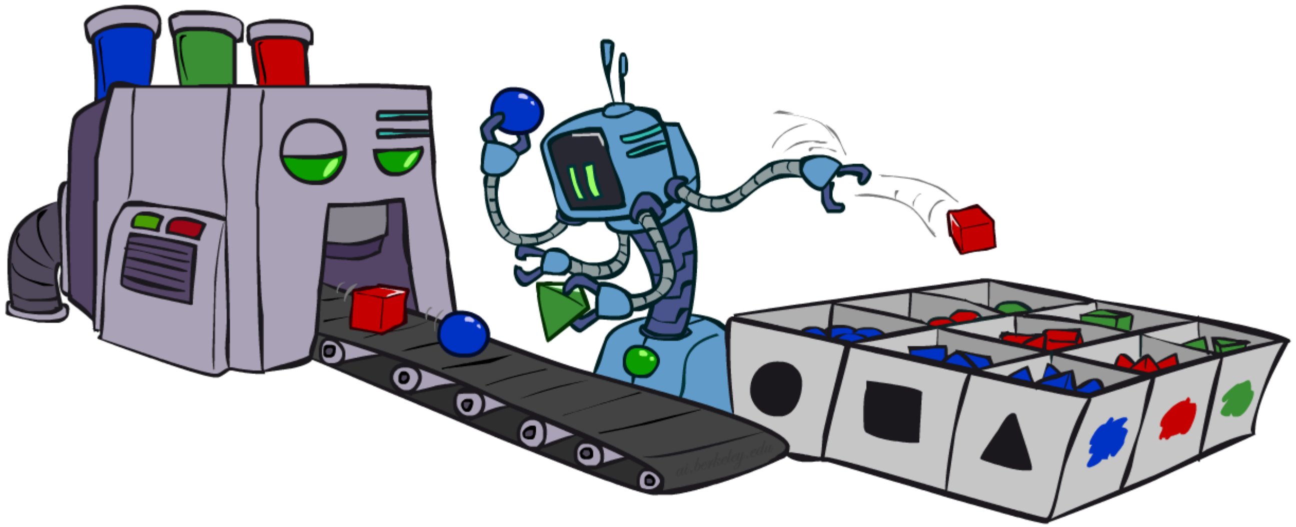
University of California, Berkeley

Sampling

- Basic idea
  - Draw *N* samples from a ***sampling distribution*** *S*
  - Compute an approximate posterior probability
  - Show this converges to the true probability *P*

- Why sample?
  - Often very fast to get a decent approximate answer
  - The algorithms are very simple and general (easy to apply to fancy models)
  - They require very little memory (O(n))
  - They can be applied to large models, whereas exact algorithms blow up

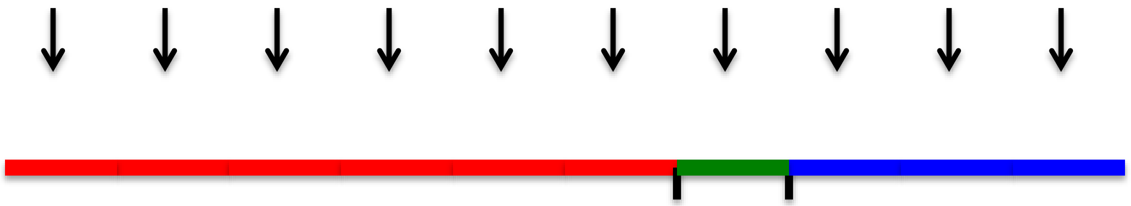


Example

- Suppose you have two agent programs ***A*** and ***B*** for Monopoly
- What is the probability that ***A*** wins?
  - Method 1:
    - Let s be a sequence of dice rolls and Chance and Community Chest cards
    - Given s, the outcome $V(s)$ is determined (1 for a win, 0 for a loss)
    - Probability that ***A*** wins is $\sum_s P(s)\ V(s)$
    - Problem: infinitely many sequences s !
  - Method 2:
    - Sample N sequences from $P(s)$, play N games (maybe 100)
    - Probability that ***A*** wins is roughly $1/N \sum_i V(s_i)$ i.e., fraction of wins in the sample

Sampling basics: discrete (*categorical*) distribution

- To simulate a biased d-sided coin:
  - Step 1: Get sample *u* from uniform distribution over [0, 1)
    - E.g. random() in python
  - Step 2: Convert this sample *u* into an outcome for the given distribution by associating each outcome *x* with a *P*(*x*)-sized sub-interval of [0,1)

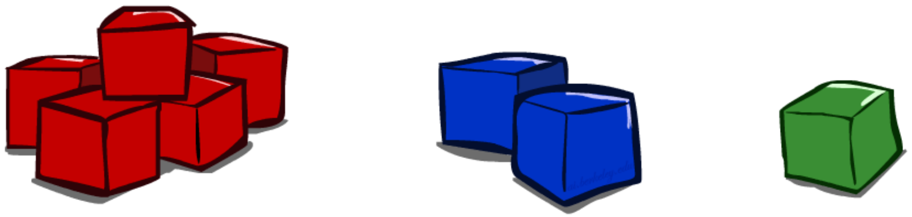


- Example

C	P(C)
red	0.6
green	0.1
blue	0.3

$0.0 \leq u < 0.6, \rightarrow C=$*red*
$0.6 \leq u < 0.7, \rightarrow C=$*green*
$0.7 \leq u < 1.0, \rightarrow C=$*blue*

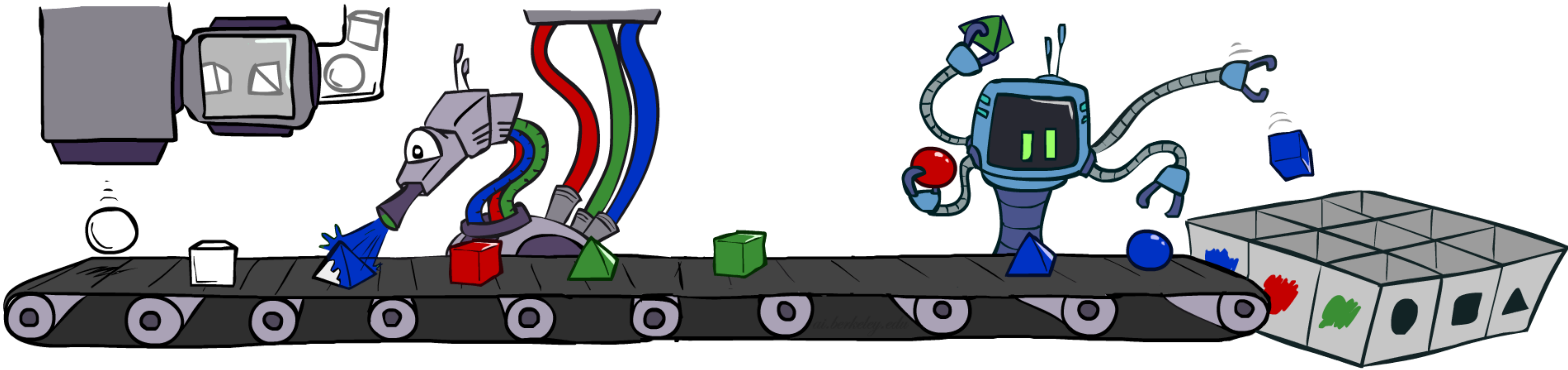
- If random() returns *u* = 0.83, then the sample is *C* = *blue*
- E.g, after sampling 8 times:



Sampling in Bayes Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

Prior Sampling



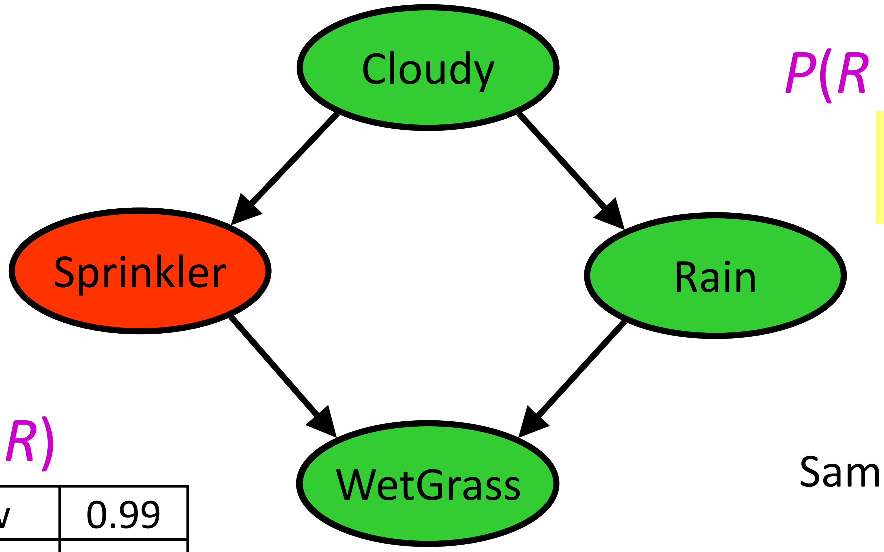
Prior Sampling

P(*C*)

c	0.5
¬c	0.5

P(*S* | *C*)

c	s	0.1
	¬s	0.9
¬c	s	0.5
	¬s	0.5



P(*R* | *C*)

c	r	0.8
	¬r	0.2
¬c	r	0.2
	¬r	0.8

P(*W* | *S*,*R*)

s	r	w	0.99
		¬w	0.01
	¬r	w	0.90
		¬w	0.10
¬s	r	w	0.90
		¬w	0.10
	¬r	w	0.01
		¬w	0.99

Samples:

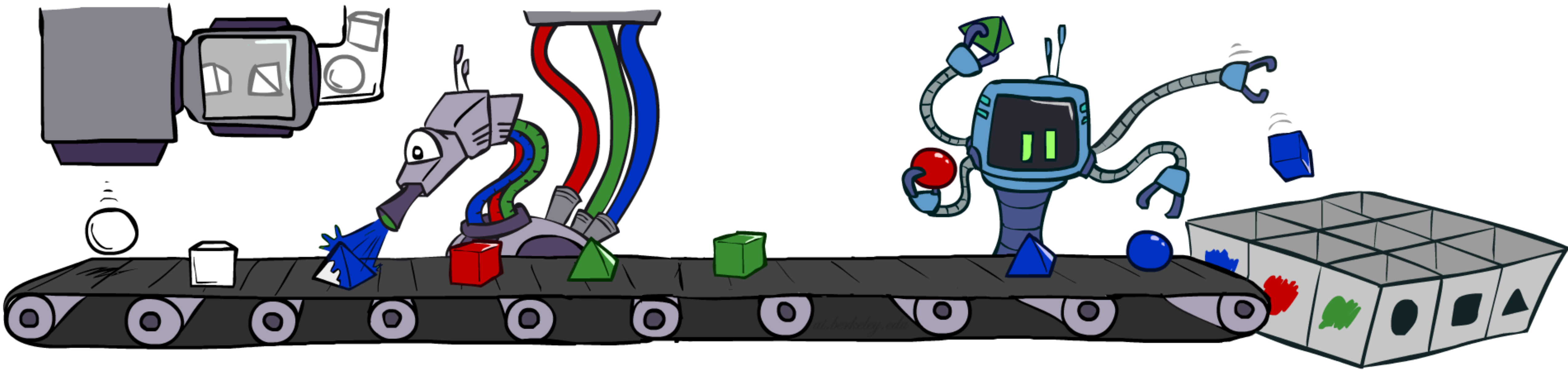
c, ¬s, r, w

¬c, s, ¬r, w

…

Prior Sampling

- For i=1, 2, …, n (in topological order)
  - Sample X_i from $P(X_i \mid parents(X_i))$
- Return $(x_1, x_2, \ldots, x_n)$



Prior Sampling

- This process generates samples with probability:

  $$S_{PS}(x_1,\ldots,x_n) = \prod_i P(x_i \mid parents(X_i)) = P(x_1,\ldots,x_n)$$

  ...i.e. the BN's joint probability

- Let the number of samples of an event be $N_{PS}(x_1,\ldots,x_n)$
- Estimate from N samples is $Q_N(x_1,\ldots,x_n) = N_{PS}(x_1,\ldots,x_n)/N$
- Then $\lim_{N\to\infty} Q_N(x_1,\ldots,x_n) = \lim_{N\to\infty} N_{PS}(x_1,\ldots,x_n)/N$

  $$= S_{PS}(x_1,\ldots,x_n)$$

  $$= P(x_1,\ldots,x_n)$$

- I.e., the sampling procedure is ***consistent***

Example

- We'll get a bunch of samples from the BN:

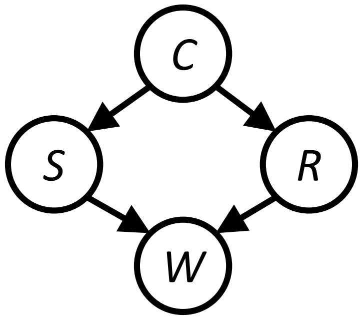
  c, ¬s, r, w

  c, s, r, w

  ¬c, s, r, ¬w

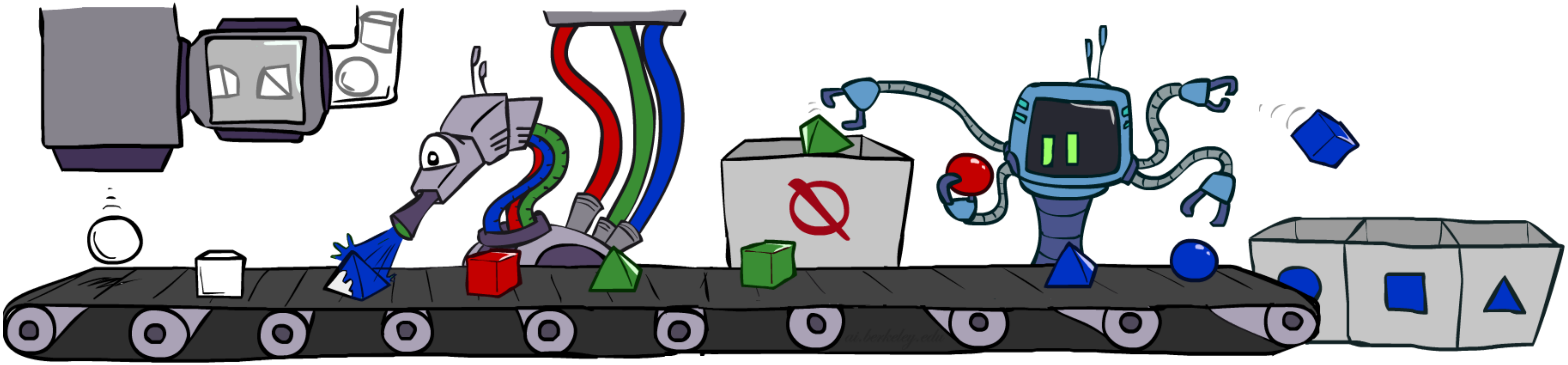
  c, ¬s, r, w

  ¬c, ¬s, ¬r, w



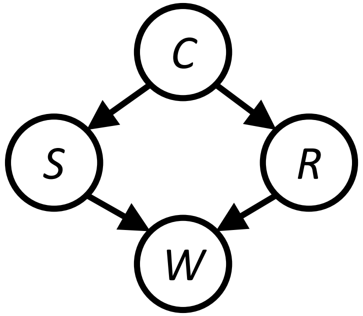
- If we want to know P(*W*)
  - We have counts <w:4, ¬w:1>
  - Normalize to get P(*W*) = <w:0.8, ¬w:0.2>
  - This will get closer to the true distribution with more samples
  - Can estimate anything else, too
    - E.g., for query $P(C \mid r, w)$ use $P(C \mid r, w) = \alpha\, P(C, r, w)$

Rejection Sampling



Rejection Sampling

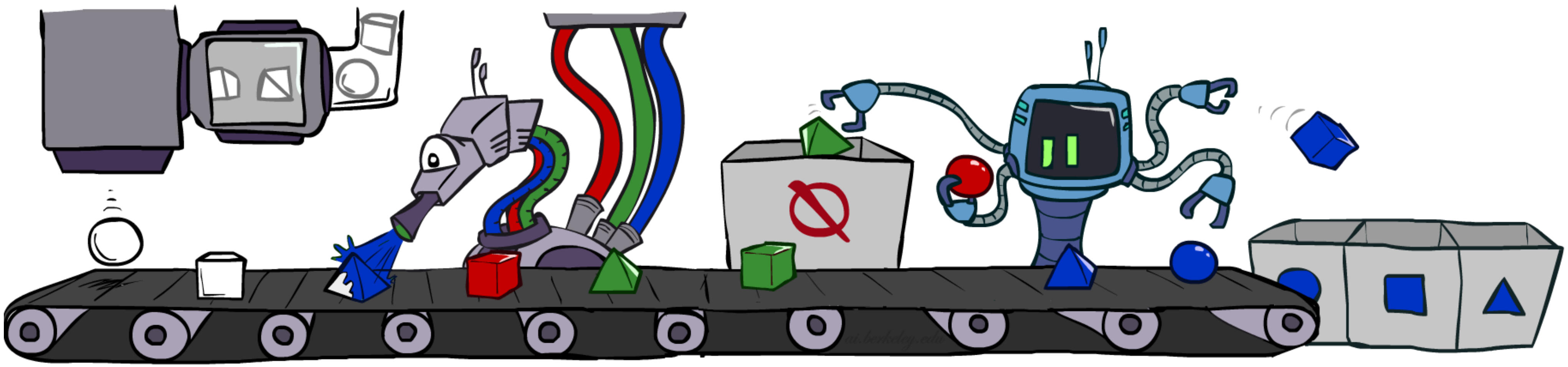
- A simple modification of prior sampling for conditional probabilities
- Let's say we want P(*C*| *r, w*)
- Count the C outcomes, but ignore (reject) samples that don't have *R*=true, *W*=true
  - This is called ***rejection sampling***
  - It is also consistent for conditional probabilities (i.e., correct in the limit)



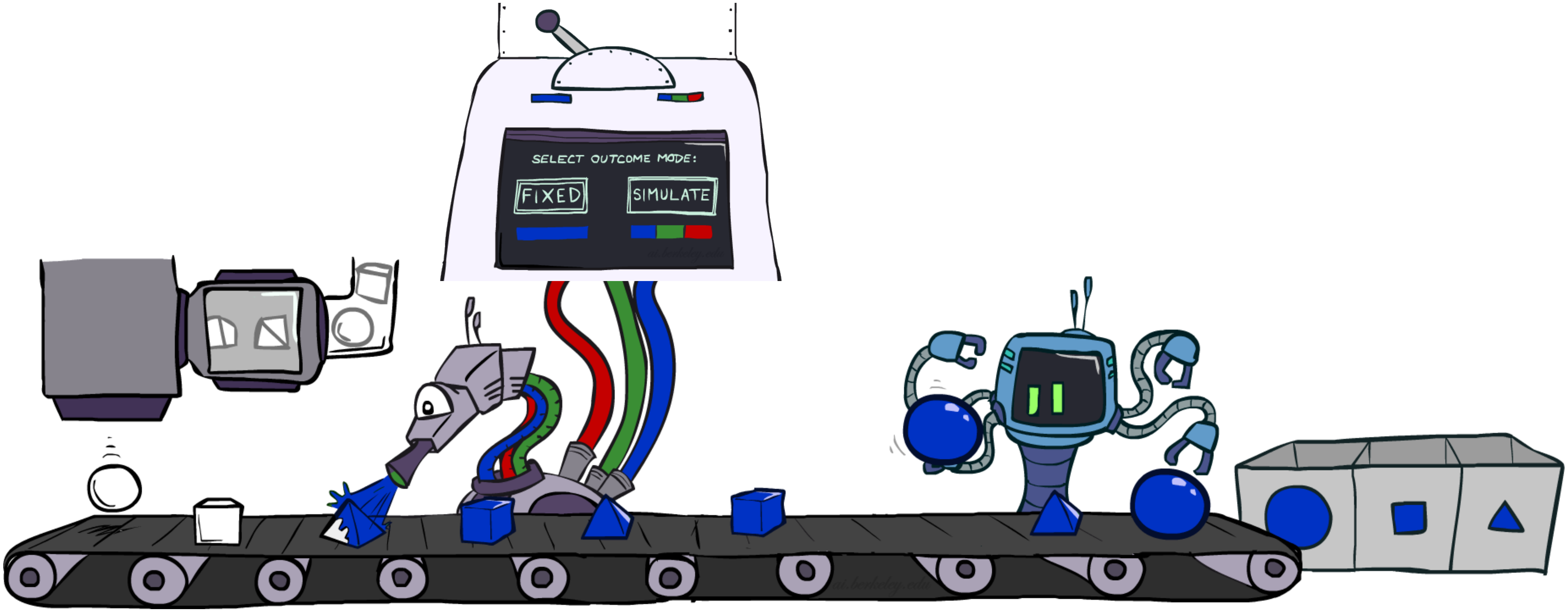
c, ¬s, r, w
~~c, s, ¬r~~
~~¬c, s, r, ¬w~~
~~c, ¬s, ¬r~~
¬c, ¬s, r, w

Rejection Sampling

- Input: evidence $e_1,..,e_k$
- For i=1, 2, …, n
  - Sample X_i from $P(X_i \mid parents(X_i))$
  - If x_i not consistent with evidence
    - Reject: Return, and no sample is generated in this cycle
- Return $(x_1, x_2, \ldots, x_n)$

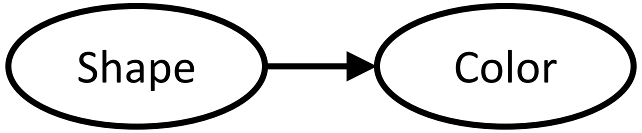


Likelihood Weighting

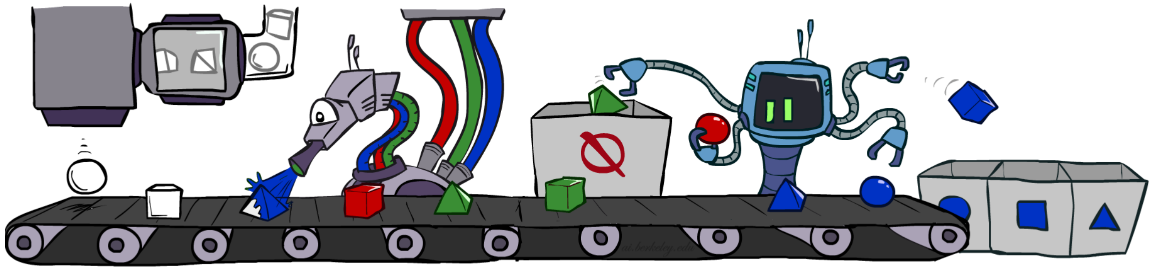


Likelihood Weighting

- Problem with rejection sampling:
  - If evidence is unlikely, rejects lots of samples
  - Evidence not exploited as you sample
  - Consider P(*Shape*|*Color=blue*)

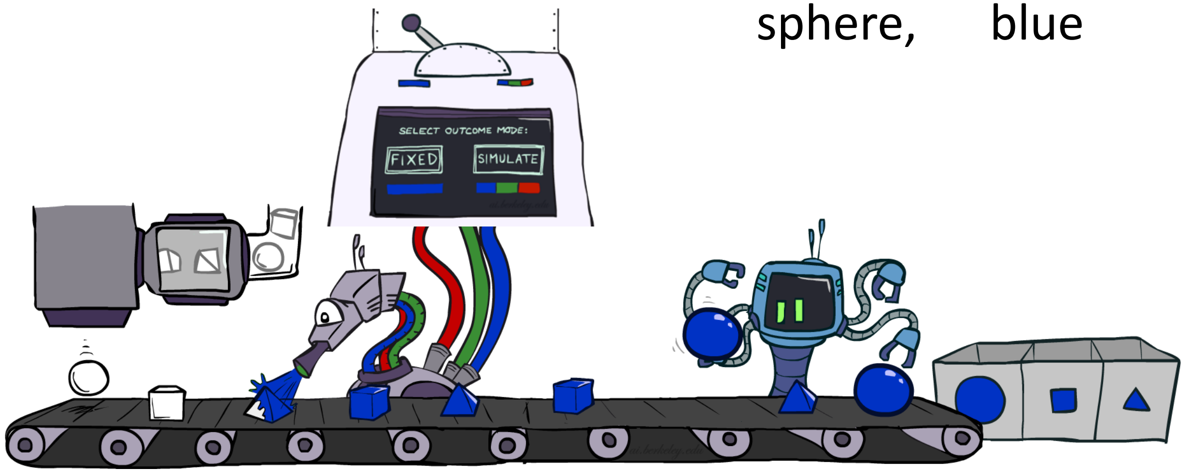


~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



- Idea: fix evidence variables, sample the rest
  - Problem: sample distribution not consistent!
  - Solution: ***weight*** each sample by probability of evidence variables given parents

pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



Likelihood Weighting

P(*C*)

c	0.5
¬c	0.5

P(*S* | *C*)

c	s	0.1
	¬s	0.9
¬c	s	0.5
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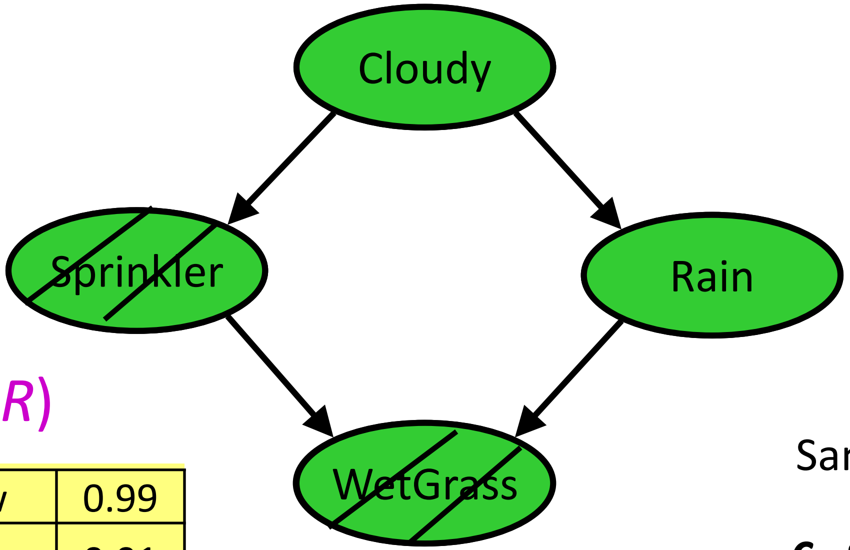


P(*R* | *C*)

c	r	0.8
	¬r	0.2
¬c	r	0.2
	¬r	0.8

P(*W* | *S*,*R*)

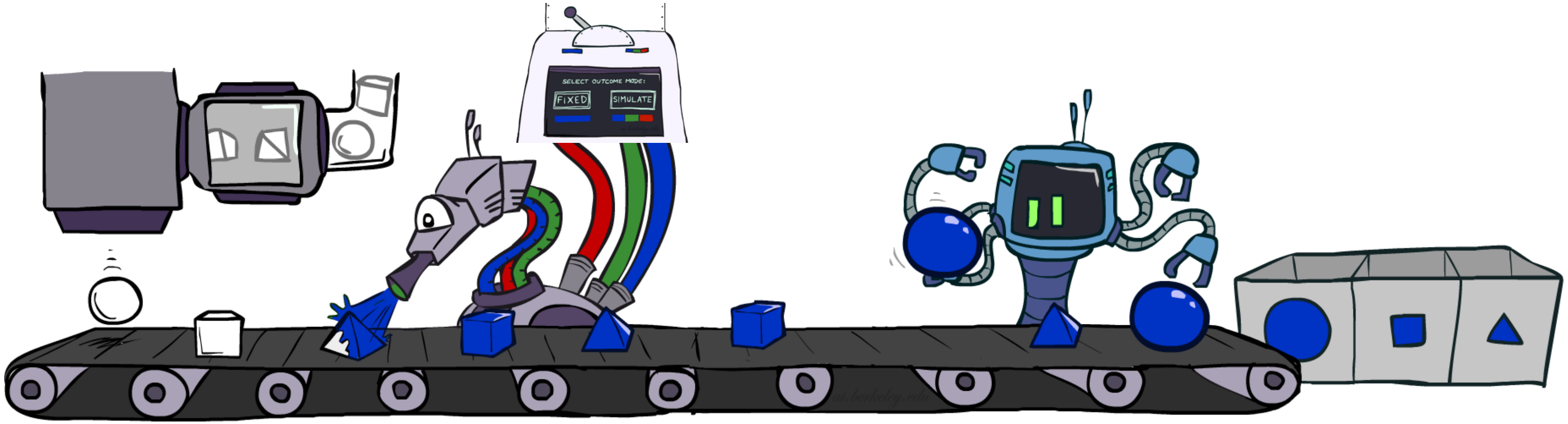
s	r	w	0.99
		¬w	0.01
	¬r	w	0.90
		¬w	0.10
¬s	r	w	0.90
		¬w	0.10
	¬r	w	0.01
		¬w	0.99

Samples:

c*, *s*, *r*, *w ***w* = 1.0 x 0.1 x 0.99**

Likelihood Weighting

- Input: evidence $e_1,..,e_k$
- $w = 1.0$
- for i=1, 2, …, n
  - if X_i is an evidence variable
    - x_i = observed value$_i$ for X_i
    - Set $w = w * P(x_i \mid Parents(X_i))$
  - else
    - Sample x_i from $P(X_i \mid Parents(X_i))$
- return $(x_1, x_2, …, x_n)$, w



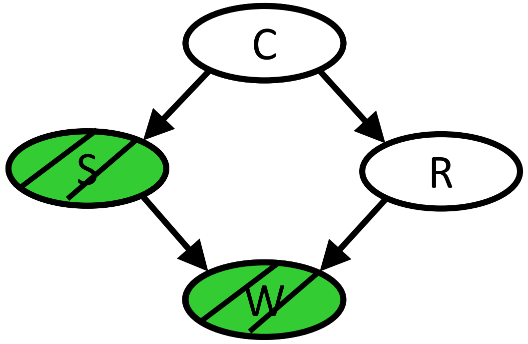
Likelihood Weighting

- Sampling distribution if **Z** sampled and **e** fixed evidence

  $$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_i P(z_i \mid parents(Z_i))$$

- Now, samples have weights

  $$w(\mathbf{z},\mathbf{e}) = \prod_j P(e_j \mid parents(E_j))$$

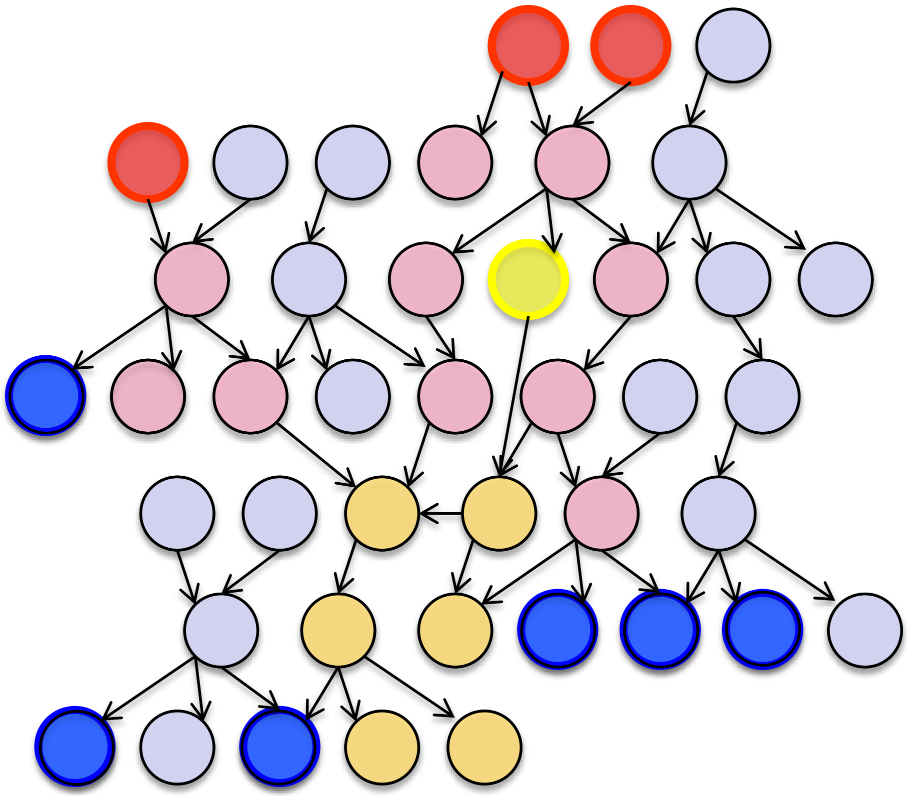


- Together, weighted sampling distribution is consistent

  $$S_{WS}(\mathbf{z},\mathbf{e}) \cdot w(\mathbf{z},\mathbf{e}) = \prod_i P(z_i \mid parents(Z_i)) \prod_j P(e_j \mid parents(E_j))$$
  $$= P(\mathbf{z},\mathbf{e})$$

Likelihood Weighting

- Likelihood weighting is good
  - All samples are used
  - The values of ***downstream*** variables are influenced by ***upstream*** evidence

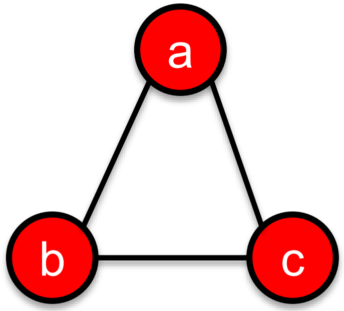


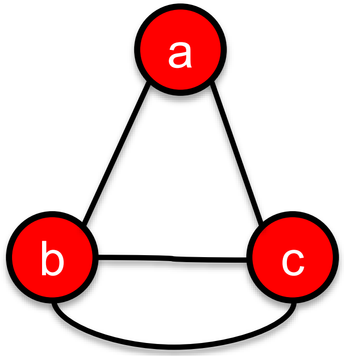
- Likelihood weighting still has weaknesses
  - The values of ***upstream*** variables are unaffected by ***downstream*** evidence
    - E.g., suppose evidence is a video of a traffic accident
  - With evidence in k leaf nodes, weights will be $O(2^{-k})$
  - With high probability, one lucky sample will have much larger weight than the others, dominating the result

- We would like each variable to "see" ***all*** the evidence!

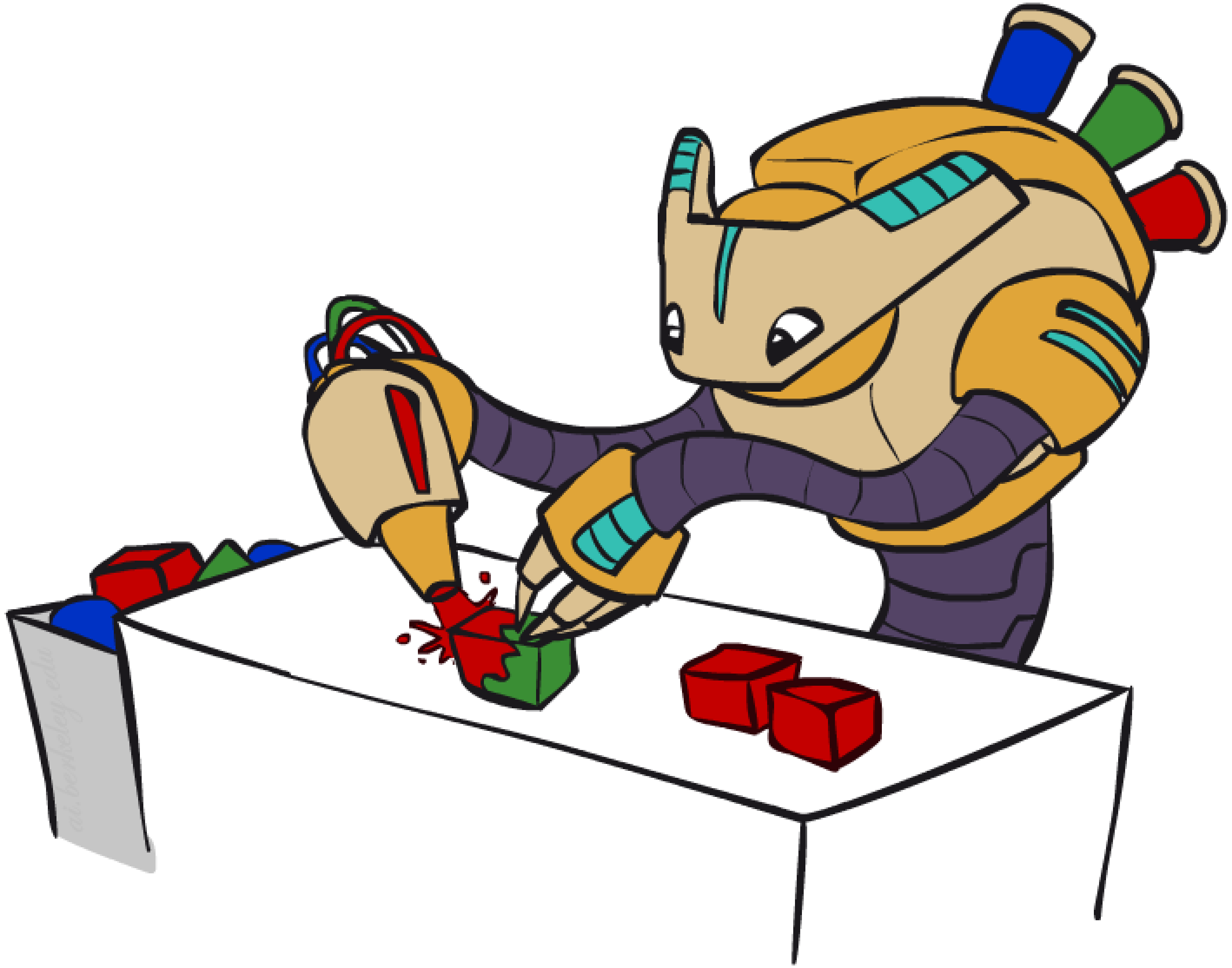
Break Quiz

- Suppose I perform a random walk on a graph, following the arcs out of a node ***uniformly at random***. In the infinite limit, what fraction of time do I spend at each node?
  - Consider these two examples:





Gibbs Sampling

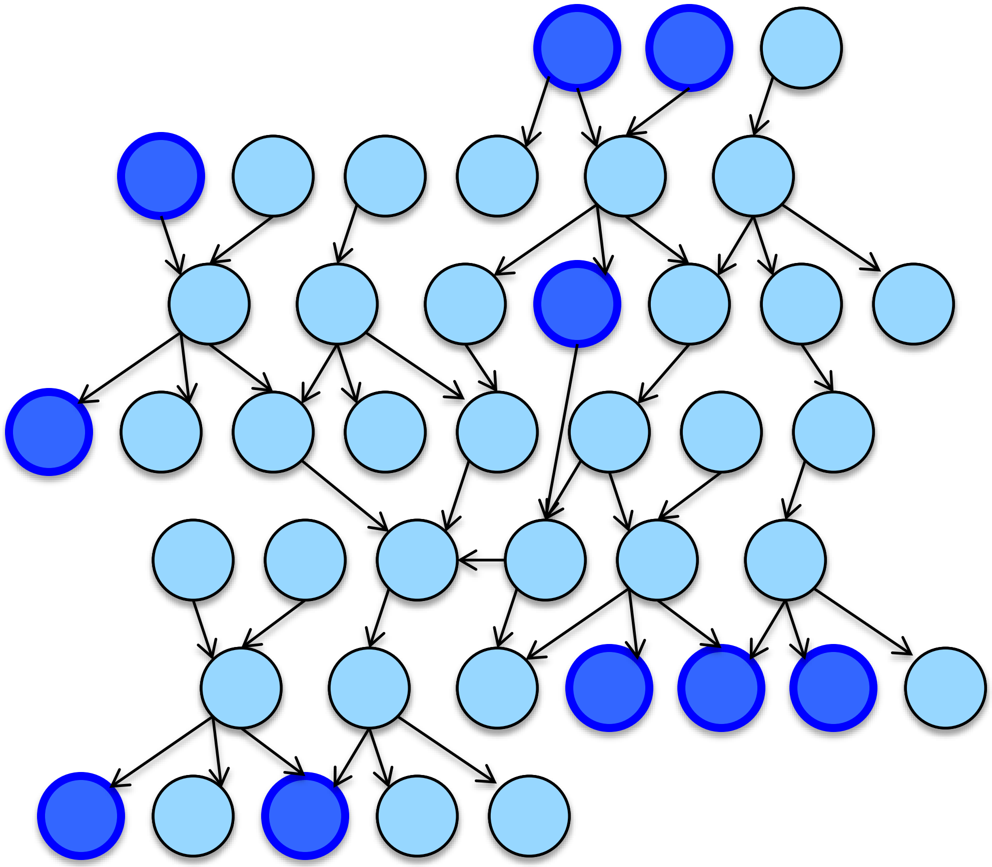


- MC… …ed algo… …r a very…
  - M… …walk”), w…
  - M… …ino
  - M… …some pr…
- MC…

Gibbs sampling

- A particular kind of MCMC
  - States are complete assignments to all variables
    - (Cf local search: closely related to min-conflicts, simulated annealing!)
  - Evidence variables remain fixed, other variables change
  - To generate the next state, pick a variable and sample a value for it conditioned on all the other variables (Cf min-conflicts!)
    - $X_i' \sim P(X_i \mid x_1,..,x_{i-1},x_{i+1},..,x_n)$
    - Will tend to move towards states of higher probability, but can go down too
    - In a Bayes net, $P(X_i \mid x_1,..,x_{i-1},x_{i+1},..,x_n) = P(X_i \mid markov\_blanket(X_i))$
- Theorem: Gibbs sampling is consistent*
  - Provided all Gibbs distributions are bounded away from 0 and 1 and variable selection is fair

Why would anyone do this?



Samples soon begin to reflect all the evidence in the network

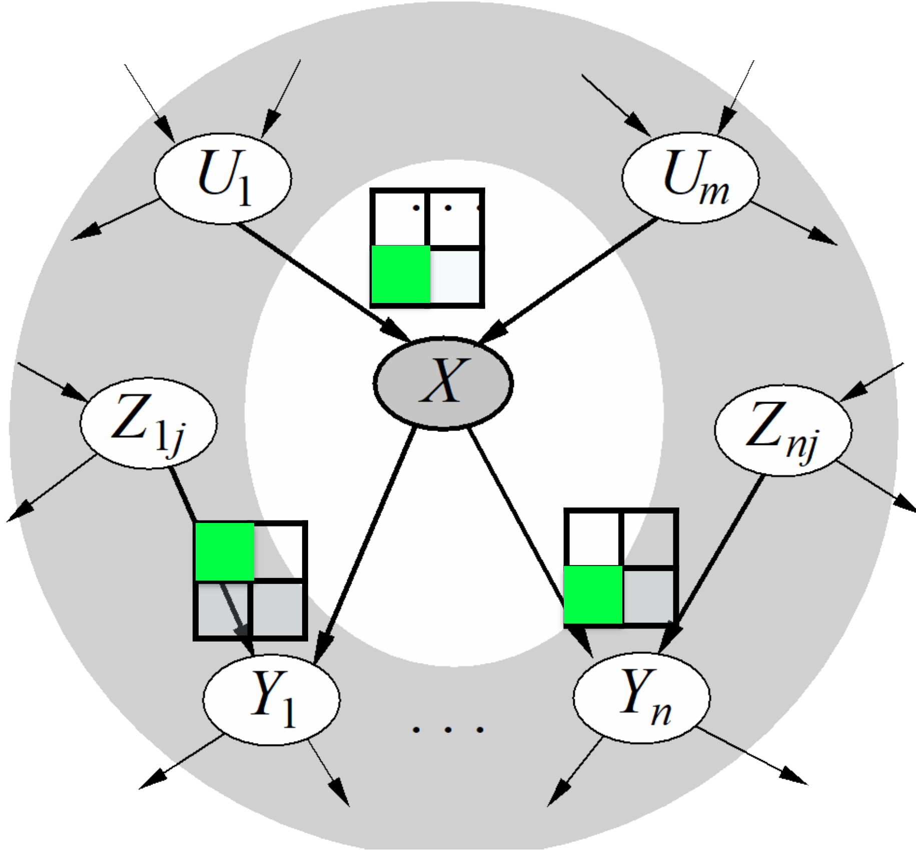
Eventually they are being drawn from the true posterior!

How would anyone do this?

- Repeat many times
  - Sample a non-evidence variable X_i from

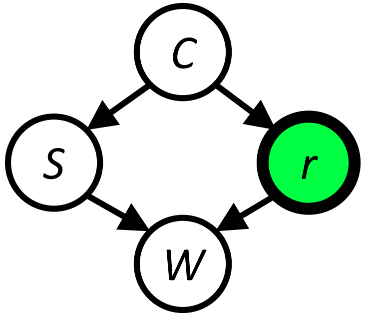
$P(X_i \mid x_1,..,x_{i-1},x_{i+1},..,x_n) = P(X_i \mid markov\_blanket(X_i))$

$= \alpha \; P(X_i \mid parents(X_i)) \prod_j P(y_j \mid parents(Y_j))$

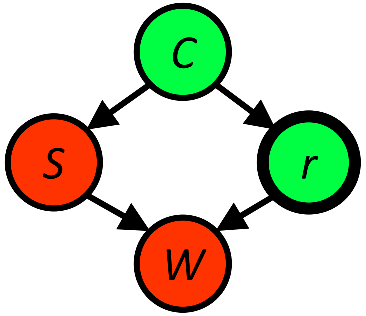


Gibbs Sampling Example: *P*(*S* | *r*)

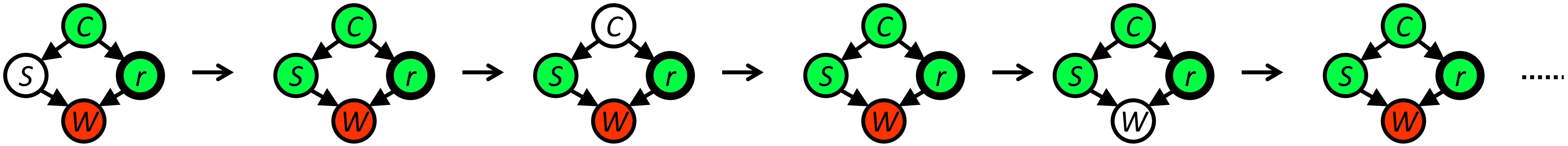
- Step 1: Fix evidence
  - *R* = true



- Step 2: Initialize other variables
  - Randomly



- Step 3: Repeat
  - Choose a non-evidence variable *X*
  - Resample *X* from *P*(*X* | *markov_blanket*(*X*))



Sample *S* ~ *P*(*S* | *c*, *r*, ¬*w*)

Sample *C* ~ *P*(*C* | *s*, *r*)

Sample *W* ~ *P*(*W* | *s*, *r*)

Why does it work? (see AIMA 14.5.2 for details)

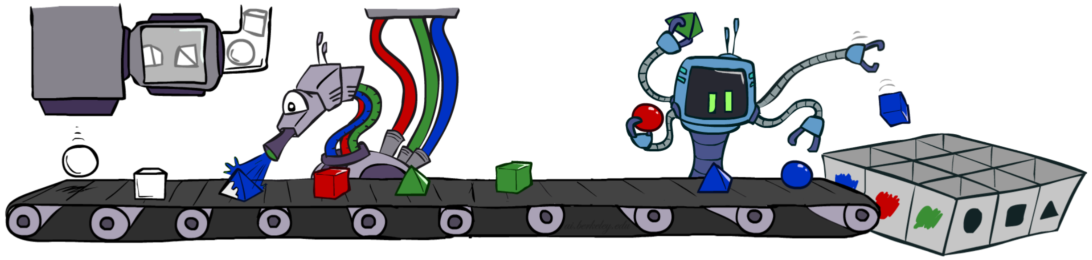
- Suppose we run it for a long time and predict the probability of reaching any given state at time t: $\pi_t(x_1,\ldots,x_n)$ or $\pi_t(\underline{\mathbf{x}})$
- Each Gibbs sampling step (pick a variable, resample its value) applied to a state $\underline{\mathbf{x}}$ has a probability $q(\underline{\mathbf{x}}' \mid \underline{\mathbf{x}})$ of reaching a next state $\underline{\mathbf{x}}'$
- So $\pi_{t+1}(\underline{\mathbf{x}}') = \sum_{\underline{\mathbf{x}}} q(\underline{\mathbf{x}}' \mid \underline{\mathbf{x}})\, \pi_t(\underline{\mathbf{x}})$ or, in matrix/vector form $\pi_{t+1} = Q\pi_t$
- When the process is in equilibrium $\pi_{t+1} = \pi_t$ so $Q\pi_t = \pi_t$
- This has a unique* solution $\pi_t = P(x_1,\ldots,x_n \mid e_1,\ldots,e_k)$
- So for large enough *t* the next sample will be drawn from the true posterior
  - "Large enough" depends on CPTs in the Bayes net; takes *longer* if nearly deterministic

Gibbs sampling and MCMC in practice

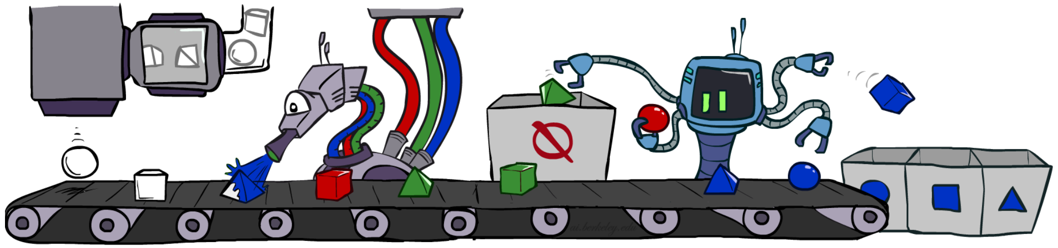
- The most commonly used method for large Bayes nets
  - See, e.g., BUGS, JAGS, STAN, infer.net, BLOG, etc.
- Can be *compiled* to run very fast
  - Eliminate all data structure references, just multiply and sample
  - ~100 million samples per second on a laptop
- Can run asynchronously in parallel (one processor per variable)
- Many cognitive scientists suggest the brain runs on MCMC

Bayes Net Sampling Summary

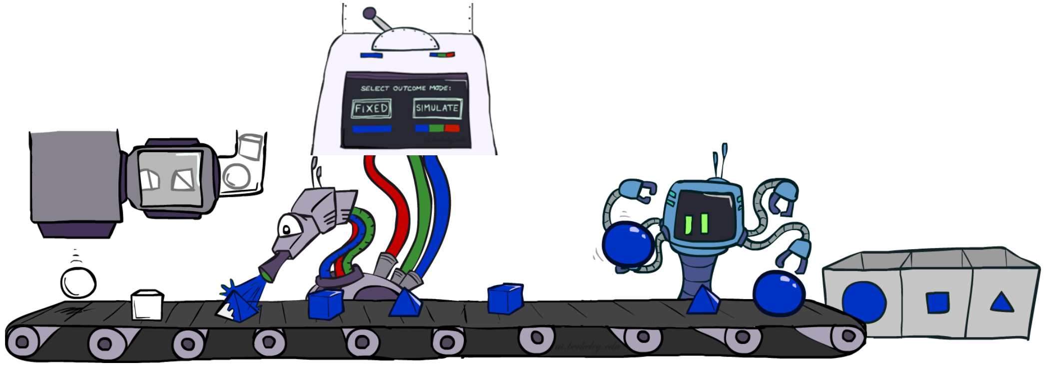
- Prior Sampling P



- Rejection Sampling P(Q | e)



- Likelihood Weighting P(Q | e)



- Gibbs Sampling P(Q | e)