C191 - Homework 1

1. **States** - Recall the states $|\pm\rangle$, defined as:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \equiv \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \equiv \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

Compute the following using both bra-ket notation and matrix notation:

(a) $\langle 0|+\rangle$
(b) $|0\rangle\langle +|$
(c) $\langle +|-\rangle$
(d) $|+\rangle\langle -|$
(e) $|+\rangle \otimes |-\rangle$

2. **Operators** - The Pauli matrices are defined as

$$\mathsf{X} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{Y} = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathsf{Z} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Write down the eigenvalues and normalized eigenstates for all three Pauli matrices.
(b) Using matrix multiplication, verify the following relations:
  i. $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathcal{I}$ where $\mathcal{I}$ is the identity matrix
  ii. $\sigma_x \sigma_y = i\sigma_z$
  iii. $[\sigma_x, \sigma_y] = 2i\sigma_z$
(c) Verify the Pauli-Euler relation,

$$\exp\left(i\theta \hat{n} \cdot \vec{\sigma}\right) = \cos\theta\, \mathcal{I} + i \sin\theta\, \hat{n} \cdot \vec{\sigma}$$

where θ is a real number, $\hat{n}$ is a unit vector in $\mathbb{R}^3$, and $\vec{\sigma}$ is the vector of Pauli matrices. Proceed by
  i. Compute $(\hat{n} \cdot \vec{\sigma})^{2m}$ for m an integer
  ii. Compute $(\hat{n} \cdot \vec{\sigma})^{2m+1}$ for m an integer
  iii. Expand the left hand side in a Taylor series about $\theta = 0$
  iv. Expand the right hand side in a Taylor series about $\theta = 0$
  v. Show that they are equal

3. **Combining quantum states** - Consider two quantum states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$
$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

Compute the following:

(a) $|\psi\rangle \otimes |\phi\rangle$
(b) $\langle \psi| \otimes \langle \phi|$
(c) $(\sigma_x \otimes \mathcal{I})(|\psi\rangle \otimes |\phi\rangle)$
(d) $(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\phi\rangle)$

4. **Entangled states -** When combining two qubits, a common basis is the *singlet-triplet* basis. The three *triplet* states are

$$\begin{aligned} |\mathsf{T}, 1\rangle &= |11\rangle \\ |\mathsf{T}, 0\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\ |\mathsf{T}, -1\rangle &= |00\rangle \end{aligned}$$

and the *singlet* state is defined as,

$$|\mathsf{S}\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

Show the following:

(a) Show that $|\mathsf{S}\rangle$ and $|\mathsf{T}, 0\rangle$ states are not separable states, $\phi \otimes \psi$, for any single-qubit states ϕ and ψ.

(b) $|\mathsf{S}\rangle$ is invariant under global rotation. That is, for any one-qubit operator U, that

$$(U \otimes U)|\mathsf{S}\rangle = |\mathsf{S}\rangle$$

5. **Dynamics -** Dynamics in quantum mechanics are given by the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

Let the Hamiltonian $H = \alpha X$ and the initial state $|\psi(0)\rangle = |0\rangle$.

(a) What is $\psi(t)$ in the $|0\rangle, |1\rangle$ basis?

(b) What is $\psi(t)$ in the $|+\rangle, |-\rangle$ basis?