Problem 1

I will first solve this problem in Dirac notation, and then I will attach a Mathematica printout where I've done the matrix algebra.

(a)

$$\langle 0|+\rangle = \langle 0|(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle) = \frac{1}{\sqrt{2}}\langle 0|1\rangle + \frac{1}{\sqrt{2}}\langle 1|1\rangle = \frac{1}{\sqrt{2}}$$

(b)

$$|0\rangle\langle +| = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1|)$$

(c)

$$\langle +|-\rangle = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 0|0\rangle) = \frac{1}{2}(1 - 0 + 0 - 1) = 0$$

(d)

$$|+\rangle\langle -| = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

(e)

$$|+\rangle \otimes |-\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

Problem 1, Matrix Form

```
plus = (1 / Sqrt[2]) (1
                      1);
minus = (1 / Sqrt[2]) ( 1
                       -1);
zero = (1
        0);
```

$$plus = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix};\quad minus = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix};\quad zero = \begin{pmatrix}1\\0\end{pmatrix};$$

(a)

In[12]:= `Transpose[zero].plus // MatrixForm`

Out[12]//MatrixForm=

$$\begin{pmatrix}\frac{1}{\sqrt{2}}\end{pmatrix}$$

(b)

In[16]:= `zero.Transpose[plus] // MatrixForm`

Out[16]//MatrixForm=

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & 0\end{pmatrix}$$

(c)

In[18]:= `Transpose[plus].minus // MatrixForm`

Out[18]//MatrixForm=

$$\begin{pmatrix}0\end{pmatrix}$$

(d)

In[19]:= `plus.Transpose[minus] // MatrixForm`

Out[19]//MatrixForm=

$$\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}$$

(e)

In[20]:= `KroneckerProduct[plus, minus] // MatrixForm`

Out[20]//MatrixForm=

$$\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$$

Problem 2

The first two parts of this I will do with Mathematica. The third part I will type up in LaTeX.

```
sx = (0 1; 1 0); sy = (0 -i; i 0); sz = (1 0; 0 -1);
```

$$sx = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ sy = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ sz = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

(a) Write down the eigenvalues and eigenstates for each Pauli matrix

Look at X first. The first eigenvector, and corresponding eigenvalue is: (Note, I added the normalization by hand because *Mathematica* doesn't do it by default)

In[39]:=
```
(1 / Sqrt[2]) Eigenvectors[sx][[1]] // MatrixForm
Eigenvalues[sx][[1]]
```

Out[39]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[40]= -1

In the above, the eigenvector is the first line and the corresponding eigenvalue is on the second. The second eigenvector, eigenvalue combination is

In[41]:=
```
(1 / Sqrt[2]) Eigenvectors[sx][[2]] // MatrixForm
Eigenvalues[sx][[2]]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[42]= 1

For Y, the combination eigenvectors and eigenvalues

In[45]:=
```
(1 / Sqrt[2]) Eigenvectors[sy][[1]] // MatrixForm
Eigenvalues[sy][[1]]
(1 / Sqrt[2]) Eigenvectors[sy][[2]] // MatrixForm
Eigenvalues[sy][[2]]
```

Out[45]//MatrixForm=

$$\begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[46]= -1

Out[47]//MatrixForm=

$$\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[48]= 1

For Z, we have

In[53]:=
```
Eigenvectors[sz][[1]] // MatrixForm
Eigenvalues[sz][[1]]
Eigenvectors[sz][[2]] // MatrixForm
Eigenvalues[sz][[2]]
```

Out[53]//MatrixForm=
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Out[54]= -1

Out[55]//MatrixForm=
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Out[56]= 1

(b) Verify some relations

Show that X^2 = Y^2 = Z^2 = I

In[58]:=
```
sx.sx // MatrixForm
sy.sy // MatrixForm
sz.sz // MatrixForm
```

Out[58]//MatrixForm=
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[59]//MatrixForm=
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[60]//MatrixForm=
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show XY = 𝕚Z

In[65]:=
```
sx.sy // MatrixForm
𝕚 * sz // MatrixForm
```

Out[65]//MatrixForm=
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Out[66]//MatrixForm=
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Show [X,Y] = 2𝕚Z. Remember, [A, B] = AB - BA

In[69]:=
```
sx.sy - sy.sx // MatrixForm
2 * 𝕚 * sz // MatrixForm
```

Out[69]//MatrixForm=
$$\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

Out[70]//MatrixForm=
$$\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

(c) Verify the Euler identity $e^{i\theta\hat{n}\cdot\vec{\sigma}} = \cos(\theta)I + i\sin(\theta)\hat{n}\cdot\vec{\sigma}$

Compute $(\hat{n}\cdot\vec{\sigma})^{2m}$

Let's look first at the base case, $m = 1$.

$$\begin{aligned}(\hat{n}\cdot\vec{\sigma})^2 &= (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 \\ &=(n_x^2+n_y^2+n_z^2)I + n_xn_y(\sigma_x\sigma_y+\sigma_y\sigma_x) + n_xn_z(\sigma_x\sigma_z+\sigma_z\sigma_x) + n_yn_z(\sigma_y\sigma_z+\sigma_z\sigma_y) \\ &=I\end{aligned}$$

Where we have used the facts that $\sigma_i^2 = I$, $n_x^2+n_y^2+n_z^2 = 1$, and $\sigma_i\sigma_j+\sigma_j\sigma_i = 0$ for $i \neq j$ (you didn't prove this earlier, but it's easy to prove to yourself with matrix multiplication). So we are left with

$$(\hat{n}\cdot\vec{\sigma})^{2m} = I$$

Compute $(\hat{n}\cdot\vec{\sigma})^{2m+1}$

Now that we have the above, this part is easy:

$$(\hat{n}\cdot\vec{\sigma})^{2m+1} = (\hat{n}\cdot\vec{\sigma})^{2m}(\hat{n}\cdot\vec{\sigma}) = \hat{n}\cdot\vec{\sigma}$$

Expand the left and right hand sides and show they are equal

Recall

$$e^x = \sum \frac{x^n}{n!}$$

$$\begin{aligned}e^{i\theta\hat{n}\cdot\vec{\sigma}} = \sum\frac{(i\theta\hat{n}\cdot\vec{\sigma})^n}{n!} &= \sum\frac{(i\theta\hat{n}\cdot\vec{\sigma})^{2m}}{(2m)!} + \sum\frac{(i\theta\hat{n}\cdot\vec{\sigma})^{2m+1}}{(2m+1)!} \\ &= \sum\frac{(i\theta)^{2m}}{(2m)!}I + \sum\frac{(i\theta)^{2m+1}}{(2m+1)!}\hat{n}\cdot\vec{\sigma} \\ &= \cos(\theta)I + i\sin(\theta)(\hat{n}\cdot\vec{\sigma})\end{aligned}$$

3. Combining quantum states

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$
$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

(a) $|\psi\rangle \otimes |\phi\rangle$

$$\begin{aligned}
|\psi\rangle \otimes |\phi\rangle &= \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}|0\rangle \otimes |0\rangle + \frac{i}{2}|0\rangle \otimes |1\rangle - \frac{i}{2}|1\rangle \otimes |0\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle \\
&= \frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{i}{2}|10\rangle + \frac{1}{2}|11\rangle
\end{aligned}$$

(b) $\langle\psi| \otimes \langle\phi|$

$$\begin{aligned}
\langle\psi| \otimes \langle\phi| &= \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{i}{\sqrt{2}}\langle 1|\right) \otimes \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right) \\
&= \frac{1}{2}\left(\langle 00| - i\langle 01| + i\langle 10| + \langle 11|\right)
\end{aligned}$$

Notice that in the first line, when going from $|\psi\rangle \to \langle\psi|$ we don't just make the kets into bras, we also apply complex conjugation of the coefficients.

(c) $(\sigma_x \otimes I)(|\psi\rangle \otimes |\phi\rangle)$

$$\begin{aligned}
(\sigma_x \otimes I)(|\psi\rangle \otimes |\phi\rangle) &= \sigma_x|\psi\rangle \otimes I|\phi\rangle \\
&= \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}(|10\rangle + i|11\rangle - i|00\rangle + |01\rangle)
\end{aligned}$$

Where we used the fact that $\sigma_x|0\rangle = |1\rangle$ and $\sigma_x|1\rangle = |0\rangle$.

(d) $(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\phi\rangle)$

$$\begin{aligned}
(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\phi\rangle) &= \sigma_x|\psi\rangle \otimes \sigma_x|\phi\rangle \\
&= \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle\right) \\
&= \frac{1}{2}(|11\rangle + i|10\rangle - i|01\rangle + |00\rangle)
\end{aligned}$$

Problem 4 Entangled states

$$\begin{aligned}
|T, 0\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\
|S\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle
\end{aligned}$$

(b) Show these two states are not separable

Suppose $|S\rangle$ were separable. Then, there would exist single qubit states $|\psi\rangle$ and $|\phi\rangle$ such that

$$|\psi\rangle \otimes |\phi\rangle = |S\rangle$$

Without loss of generality, let

$$\begin{aligned} |\psi\rangle =& a|0\rangle + b|1\rangle \\ |\phi\rangle =& c|0\rangle + d|1\rangle \end{aligned}$$

for some undetermined coefficients. Then:

$$|\psi\rangle \otimes |\phi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

But since this must equal $|S\rangle$, we must have $ac = 0$ and $bd = 0$ since $|S\rangle$ has no component of $|00\rangle$ or $|11\rangle$. Then either $a = 0$ or $c = 0$. If $a = 0$, we have

$$|\psi\rangle \otimes |\phi\rangle = bc|10\rangle + bd|11\rangle$$

If $c = 0$, we have

$$|\psi\rangle \otimes |\phi\rangle = ad|01\rangle + bd|11\rangle$$

It is now clear by inspection that in either case, no matter the choice of the remaining coefficients, we cannot make this equal to $|S\rangle$. This proof holds without modification for $|T, 0\rangle$.

(b) Show $|S\rangle$ is invariant under global rotation

For any single qubit operator U, let

$$\begin{aligned} U|0\rangle =& a|0\rangle + b|1\rangle \\ U|1\rangle =& c|0\rangle + d|1\rangle \end{aligned}$$

Now:

$$\begin{aligned} (U \otimes U)|S\rangle =& \frac{1}{\sqrt{2}}(U \otimes U)(|01\rangle - |10\rangle) \\ =& \frac{1}{\sqrt{2}}((a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) - (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle)) \\ =& \frac{1}{\sqrt{2}}(ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle - ac|00\rangle - bc|01\rangle - ad|10\rangle - bd|11\rangle) \\ =& \frac{1}{\sqrt{2}}((ad - bc)|01\rangle - (ad - bc)|10\rangle \\ =& \frac{ad - bc}{\sqrt{2}}(|01\rangle - |10\rangle) \\ =& (ad - bc)|S\rangle \end{aligned}$$

We are almost there! Remember that we require that all quantum states be normalized to 1. (More formally, $U \otimes U$ is a unitary operator, one property of which is that it preserves the norm of vectors it acts on.) This means that $|ad - bc|^2 = 1$. This means that $ad - bc = e^{i\theta}$ for some θ. Thus, our final state is $e^{i\theta}|S\rangle$, which is just an overall phase times $|S\rangle$. In quantum mechanics, we don't care about overall phases acting on our state vectors, so we make the identification $e^{i\theta}|S\rangle \sim |S\rangle$.

Problem 5 Dynamics

(a) The $|0\rangle, |1\rangle$ basis

Let $|\psi(t)\rangle = a|0\rangle + b|1\rangle$. Then $H|\psi(t)\rangle = \alpha(a|1\rangle + b|0\rangle)$. $\frac{\partial}{\partial t}|\psi(t)\rangle = \dot{a}|0\rangle + \dot{b}|1\rangle$. Here, the dots represent differentiation with respect to time. So our differential equation reads

$$i\hbar(\dot{a}|0\rangle + \dot{b}|1\rangle) = \alpha(a|1\rangle + b|0\rangle)$$

We can set the coefficients of each vector equal on each side of the equation, yielding the following coupled equations

$$\begin{aligned}\frac{\partial a}{\partial t} &= -i\frac{\alpha}{\hbar}b \\ \frac{\partial b}{\partial t} &= -i\frac{\alpha}{\hbar}a\end{aligned}$$

Thus, recognize that the differential equation for the coefficients is a matrix equation of the form

$$\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\alpha}{\hbar}\sigma_x\begin{pmatrix} a \\ b \end{pmatrix}$$

Recall that the solution to such an equation is given by:

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = e^{i\alpha t/\hbar\sigma_x}\begin{pmatrix} a(0) \\ b(0) \end{pmatrix}$$

Now we can use the result from Problem 2, and write

$$e^{i\alpha t/\hbar\sigma_x} = \cos(\alpha t/\hbar)I + i\sin(\alpha t/\hbar)\sigma_x$$

. Plugging in our initial condition $(a(0) = 1, b(0) = 0)$, we have

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = (\cos(\alpha t/\hbar)I + i\sin(\alpha t/\hbar)\sigma_x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Or

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t/\hbar) \\ i\sin(\alpha t/\hbar) \end{pmatrix}$$

We can, if you'd like, go back to Dirac notation and write

$$|\psi(t)\rangle = \cos(\alpha t/\hbar)|0\rangle + i\sin(\alpha t/\hbar)|1\rangle$$

(b) The $|+\rangle, |-\rangle$ basis

If we wanted, we could rewrite the Schrodinger equation in the the $|+\rangle, |-\rangle$ basis, but since we've already done the work of solving, let's just take the answer from the previous problem and write it in the new basis.

$$\begin{aligned}|0\rangle &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \\ |1\rangle &= \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\end{aligned}$$

Plugging in to the solution from part (a), we get

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{\sqrt{2}}(\cos(\alpha t/\hbar) + i\sin(\alpha t/\hbar))|+\rangle + \frac{1}{\sqrt{2}}(\cos(\alpha t/\hbar) - i\sin(\alpha t/\hbar))|-\rangle \\ &= \frac{1}{\sqrt{2}}e^{i\alpha t/\hbar}|+\rangle + \frac{1}{\sqrt{2}}e^{-i\alpha t/\hbar}|-\rangle\end{aligned}$$