C191 - Homework 2

1. **The secular approximation -** The *hyperfine contact interaction* in Hydrogen governs the interaction between the electron spin and the nuclear proton spin. This interaction is represented by the Hamiltonian,

$$H_0 = J\vec{\sigma}^{(e)} \cdot \vec{\sigma}^{(p)} \tag{1}$$
$$= J\left(\sigma_x^{(e)} \otimes \sigma_x^{(p)} + \sigma_y^{(e)} \otimes \sigma_y^{(p)} + \sigma_z^{(e)} \otimes \sigma_z^{(p)}\right) \tag{2}$$
$$= J\left(2\sigma_+^{(e)} \otimes \sigma_-^{(p)} + 2\sigma_-^{(e)} \otimes \sigma_+^{(p)} + \sigma_z^{(e)} \otimes \sigma_z^{(p)}\right) \tag{3}$$

Where the superscripts $^{(e)}$ and $^{(p)}$ indicate that the operator acts on the electron or proton, respectively, and the spin raising/lowering matrices are:

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Suppose now that a large magnetic field, B, is applied, interacting with the spins via the following Hamiltonian,

$$H_1 = \frac{1}{2} g\mu_B B\sigma_z^{(e)} - \frac{1}{2} g\mu_n B\sigma_z^{(p)}$$

Notice that the sign and magnitude of the interaction is different for the electron and the proton: the nuclear magneton, $\mu_n = e\hbar/2m_p$, is roughly 2000 times smaller than the Bohr magneton, $\mu_B = e\hbar/2m_e$, because the ratio of the proton mass to the electron mass is $m_p/m_e \simeq 2000$, while the sign difference is due to the opposite signs of the electron and proton charge. The total Hamiltonian is then $H = H_0 + H_1$. Show that, if the magnetic field is sufficiently large, then we can take the *secular approximation*,

$$H \simeq H_{\text{sec}} = J\sigma_z^{(e)}\sigma_z^{(p)} + \frac{1}{2} g\mu_B B\sigma_z^e - \frac{1}{2} g\mu_n B\sigma_z^p$$

To do this,

   (a) Transform into a rotating frame with respect to H_1. That is, determine the Hamiltonian governing the evolution of the state

   $$|\psi'(t)\rangle = \exp\left(iH_1 t/\hbar\right)|\psi(t)\rangle$$

   Where $|\psi(t)\rangle$ obeys the Schrodinger equation:

   $$i\hbar\frac{d}{dt}|\psi(t)\rangle = (H_0 + H_1)|\psi(t)\rangle$$

   It will help to prove the following identities:

   $$\exp\left(i\alpha\sigma_z\right)\sigma_+\exp\left(-i\alpha\sigma_z\right) = e^{2i\alpha}\sigma_+$$

   $$\exp\left(i\alpha\sigma_z\right)\sigma_-\exp\left(-i\alpha\sigma_z\right) = e^{-2i\alpha}\sigma_-$$

   (b) Argue that all quickly oscillating terms may be neglected. Why does the magnetic field need to be large?

   (c) Transform back into the lab frame.

2. **Spin echo -** Suppose that a qubit, initially in the state $|+\rangle$, is subjected to a magnetic field of unknown strength, B, experiencing a Hamiltonian,

$$H = KB\sigma_z$$

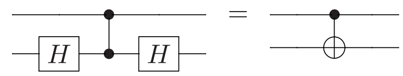
   (a) What is the state of the qubit after a time, τ?

   (b) At time τ, a strong field is applied, causing a unitary operator, σ_x to be applied to the qubit. What is the state of the qubit now?

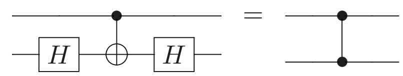
(c) The qubit is allowed to evolve under the Hamiltonian H for an additional time τ. What is the final state of the qubit?

3. **Circuit model exercises**

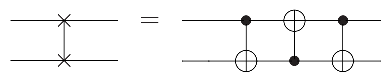
(a) Show that a CNOT can be achieved from a CPHASE and two Hadamard operations



(b) Show the reverse,

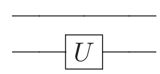


(c) Show that a SWAP gate may be implemented as three CNOTs



(d) For a matrix $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, what is the matrix representation of the following quantum circuits

i.



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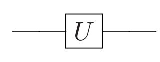

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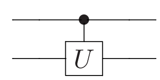

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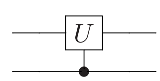

4. **Single qubit gates** We saw in class that the Hamiltonian,

$$H = -\frac{1}{2} g\mu_B \left(B_0\sigma_z + B_1\cos(\omega t)\sigma_x\right)$$

could generate a unitary operator $U(\tau) \propto \sigma_x$ if applied for a time, τ. Now let's change the phase on the oscillating term,

$$H_1 = -\frac{1}{2} g\mu_B \left(B_0\sigma_z + B_1\sin(\omega t)\sigma_x\right)$$

(a) If this Hamiltonian is allowed to act for the same time, τ, what unitary operator is generated (in the rotating frame)?

(b) If we wanted to generate the unitary operator,

$$U \propto \cos(\theta)\sigma_x + \sin(\theta)\sigma_y,$$

what Hamiltonian could we apply to do this? You should check that this operator is, in fact, unitary!