C191 - Homework 2

1. **The Chernoff Bound -** The Chernoff bound is perhaps the most important of several *tail inequalities* bounding the probabilities of rare events. Once version of the bound can be stated as follows:

   Consider a set of independent and identically distributed (*iid*) random variables, $X_1, X_2, \ldots, X_n$, each taking a value $0 \leq X_i \leq 1$. Define $X = \sum_{i=1}^{n} X_i$, the sum of the random variables. Because the sum is itself a random variable, we can define its expectation value $\mu = \langle X \rangle = n \langle X_i \rangle$. The Chernoff bound states that, for any $\epsilon \geq 0$, the probability that the sum is far from its expectation value, μ, decays exponentially. Specifically,

   $$\Pr[X \geq (1+\epsilon)\mu] \leq \exp\left(-\frac{\epsilon^2}{2+\epsilon}\mu\right)$$

   and

   $$\Pr[X \leq (1-\epsilon)\mu] \leq \exp\left(-\frac{\epsilon^2}{2}\mu\right).$$

   It is often helpful to use the two-sided version of this bound,

   $$\Pr[|X-\mu| \geq \epsilon\mu] \leq 2\exp\left(-\frac{\epsilon^2}{2+\epsilon}\mu\right)$$

   (a) Suppose we have a biased coin that comes out heads with probability 2/3 and tails with probability 1/3. Suppose the coin is flipped n times.

      i. Use the Chernoff bound to estimate how large n must be so that the probability of less than half the flips coming out heads is smaller than $10^{-}4$.

      ii. Repeat the above calculation but solve it exactly.

   (b) (Not required) Now suppose that we are conducting a poll consisting of a single yes/no question, to which some fraction of the population, p, would answer "yes." We will poll n people and use the sample average, $\bar{X}$ as our estimate of p, where $\bar{X}$ is the fraction of respondents who report "yes". We want to know: how large does n have to be so that we have an accuracy, θ, and a confidence, $1-\delta$, meaning,

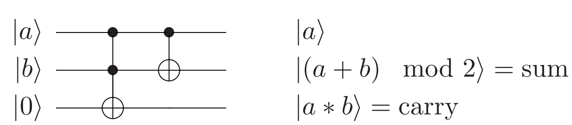
   $$\Pr[\left|\bar{X}-p\right| \leq \theta] \geq 1-\delta$$

   Using the Chernoff bound, show that to satisfy the above criterion, we must have

   $$n \geq \frac{2+\epsilon}{\epsilon^2}\ln\left(\frac{2}{\delta}\right)$$

   This result is known as the *Sampling Theorem* and arises frequently when measuring the parameters of a physical system.

2. **Quantum half-adder -** The following circuit, which you'll recognize from the midterm, is known as the *quantum half-adder*.

   This circuit takes two one-bit numbers (plus an ancilla bit) and adds them; returning their sum, the carry, and another bit required to make the circuit reversible:

   $$|a\rangle|b\rangle \otimes |0\rangle \rightarrow |a*b\rangle |(a+b) \mod 2\rangle \otimes |a\rangle$$

   Adding two two-bit numbers should therefore return a state of the form

   $$|Aa\rangle |Bb\rangle \otimes |0\rangle^{\otimes n} \rightarrow |(A*B + A*a*b + B*a*b)\rangle |(A+B+a*b)\rangle |a+b\rangle \otimes |\psi\rangle$$

   where all additions are taken $\mod 2$ and $|\psi\rangle$ is the state of the remaining ancilla after the addition.

(a) Show that, for addition of two two-bit numbers,

$$Aa + Bb = (A * B + A * a * b + B * a * b)(A + B + a * b)(a + b)$$

where the bits on the RHS are separated by parentheses.

(b) Using several copies of the half-adder circuit, construct a circuit which adds two two-bit numbers.

(c) How many total qubits do you need?

(d) Verify that your circuit works by tracking the state through each adder as the system adds 10 (binary 2) to 11 (binary 3).

(e) What is the output of the circuit?

3. **Universality** - Suppose we can only apply the following two one-qubit gates,

$$U_X = \exp(i\epsilon X)$$

$$U_Y = \exp(i\epsilon Y)$$

as well as their inverses, $U_X^\dagger, U_Y^\dagger$, for some small angle ϵ.

(a) Show that

$$U_Y^\dagger U_X^\dagger U_Y U_X = \exp\left(2i\epsilon^2 Z + \mathcal{O}\left(\epsilon^3\right)\right)$$

(b) How would you use these three unitary operators to approximate the gate

$$W = \exp(-i\epsilon(10X + 20Y + 10\epsilon Z))$$

(c) (Not required) What is the error in your approximation?

4. **Measurement** - Suppose we wished to measure the operator $\hat{n} \cdot \vec{\sigma}$ on a qubit that is initially in the state $|0\rangle$.

(a) What are the corresponding measurement operators?

(b) In terms of the spherical polar angles, θ, ϕ, associated with $\hat{n}$, what is the probability of obtaining the result $+1$?

(c) What is the state after the measurement?