CS191 – Fall 2014
Homework 4: due in lecture Oct. 15th

1. ***Fidelity calculation.*** Consider the mixed state

$$\rho = p\frac{I}{d} + (1-p)|\psi\rangle\langle\psi|,$$

where $0 \le p \le 1$ and d is the dimension of the system (*e.g.*, $d = 2$ is this is the state of a qubit). As we will learn later in the course, this is output state of a *depolarizing* process, which is a model of a noisy channel, when the input is the pure state $|\psi\rangle$. You can think of this process as replacing the input state with the completely mixed state (I/d) with some probability p.

   (a) Calculate the fidelity between the output of the depolarizing process and its input. That is, calculate $F(\rho, |\psi\rangle\langle\psi|)$.

   (b) For fixed p, does this fidelity get larger or smaller as the dimension d increases? Interpreting fidelity as a distance in Hilbert space, can you provide some geometric intuition for this behavior with d?

2. ***Partial trace calculation.*** Let the initial state of two qubits be $|00\rangle$. Then we apply a two-qubit unitary $U(\theta) = \exp(-i\frac{\theta}{2}\sigma_x \otimes \sigma_x)$, for $0 \le \theta \le 2\pi$, to this state to get:

$$|\psi(\theta)\rangle = U(\theta)|00\rangle$$

   (a) The resulting state will take the form $|\psi(\theta)\rangle = a(\theta)|00\rangle + b(\theta)|01\rangle + c(\theta)|10\rangle + d(\theta)|11\rangle$. Calculate the coefficients $a(\theta), b(\theta), c(\theta), d(\theta)$.

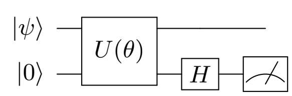
   Hint: $\exp(-i\theta A) = \cos(\theta)I - i\sin(\theta)A$ for any operator A that squares to the identity.

   (b) Write the reduced state of the first qubit as a function of θ after tracing out the second qubit.

   (c) For what value(s) of θ is the reduced state of the first qubit a pure state?

3. ***Generalized measurement.*** In the lecture notes we went through how a generalized measurement can be implemented by coupling to an ancilla and then doing projective measurements on the ancilla degrees of freedom. In this problem you will work out what the POVM elements are for a particular implementation of a type of generalized measurement called a *weak measurement*.

Let the main system be a qubit in an arbitrary state $|\psi\rangle$. This qubit is coupled to another qubit that constitutes the ancilla, according to the circuit



The projective measurement of the ancilla qubit is in the computational basis, and the H represents a Hadamard gate. The coupling unitary has the form:

$$U(\theta) = e^{-i\frac{\theta}{2}Z\otimes Y}, \qquad 0 \le \theta \le 2\pi$$

   (a) Write down the state of the two qubits just before the measurement. That is, compute the effect of the two gates in the circuit. You might find the hint to question 2(a) useful here also.

   (b) The projective measurement of the ancilla qubit in its computational basis has two possible outcomes $|0\rangle, |1\rangle$. Calculate the probability of each of these outcomes, using the expression for the pre-measurement state you computed in part (a).

   (c) In the lecture notes, we defined the generalized measurement operator on the system corresponding to an outcome i on the ancilla as

$$M_i|\psi\rangle_S \equiv \langle m_i|U_{SA}(|\psi\rangle_S \otimes |\phi\rangle_A),$$

   where U_{SA} is the coupling between the system and ancilla, and $|\phi\rangle_A$ the initial state of the ancilla. You computed $U_{SA}|\psi\rangle_S \otimes |\phi\rangle_A$ in part (a). And $\langle m_i| \in \{_A\langle 0|, _A\langle 1|\}$. Using this, work out the two possible measurement operators, M_0 and M_1.

(d) Compute the two $E_i = M_i^\dagger M_i$ POVM elements, and confirm that $\sum_i E_i = I$.

(e) Write each measurement operator M_i in the form:

$$M_i = a_i|0\rangle\langle 0| + b_i|1\rangle\langle 1|,$$

for some θ dependent coefficients a_i, b_i. By doing this notice the following:

i. These measurement operators are not projectors (rank 1) in general.

ii. When $\theta = 0$ both $M_i = I$, meaning that there is no measurement of the system state. This makes sense because $U(\theta = 0) = I$, and there is no interaction between the system and ancilla.

iii. When $\theta = \pi/2$, the measurement operators are projectors onto the computational basis states, and thus this generalized measurement becomes a von Neumann projective measurement.

iv. When θ is between these two values, this circuit implements a *weak measurement* of the system, where one gains a little bit of information about the system at the expense of only a little disturbance on the system.

4. ***CHSH inequality.*** Consider the CHSH game that we discussed in class, with the following choices for Alice's and Bob's observables:

$$\begin{aligned}
P &: \quad \sigma_z^A \\
Q &: \quad \cos(\frac{\pi}{4})\sigma_z^A + \sin(\frac{\pi}{4})\sigma_x^A \\
R &: \quad \sigma_z^B \\
S &: \quad \cos(\frac{\pi}{4})\sigma_z^B - \sin(\frac{\pi}{4})\sigma_x^B,
\end{aligned}$$

where $\sigma_z^A \equiv \sigma_z^A \otimes I^B$ is an observable on Alice's qubit only, and so on. Let the two-qubit state shared by Alice and Bob be an imperfect entangled state:

$$\rho = p\frac{I}{4} + (1-p)\left|\psi^{AB}\right\rangle\left\langle\psi^{AB}\right|,$$

where $\left|\psi^{AB}\right\rangle = \frac{1}{\sqrt{2}}(\left|0^A\right\rangle\left|1^B\right\rangle - \left|1^A\right\rangle\left|0^B\right\rangle)$. Calculate the CHSH quantity:

$$|CHSH| \equiv |E(P,R) + E(Q,R) + E(P,S) - E(Q,S)|,$$

for this state, as a function of p. For what values of p is the CHSH inequality violated?