CS191 – Fall 2014
Homework 4 solutions

1. ***Fidelity calculation.***

$$\rho = p\frac{I}{d} + (1-p)|\psi\rangle\langle\psi|$$

   (a) $F(\rho, |\psi\rangle\langle\psi|) = \sqrt{\langle\psi|\rho|\psi\rangle} = \sqrt{\frac{p}{d} + 1 - p} = \sqrt{1 - (p - \frac{p}{d})}$

   (b) For fixed p this fidelity decreases as d increases. Intuitively this is because as the dimension of the state space increases, there are "more directions to move in". Therefore, adding some portion, p, of the identity matrix perturbs the state a greater distance from the original pure state.

2. ***Partial trace calculation.***
   $|\psi(\theta)\rangle = U(\theta)|00\rangle$, with $U(\theta) = \exp(-i\frac{\theta}{2}\sigma_x \otimes \sigma_x)$

   (a) $U(\theta) = \exp(-i\frac{\theta}{2}\sigma_x \otimes \sigma_x) = cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})\sigma_x\sigma_x$. Therefore,

$$|\psi(\theta)\rangle = \cos(\frac{\theta}{2})|00\rangle - i\sin(\frac{\theta}{2})|11\rangle,$$

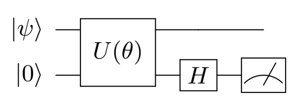
   whence, $a(\theta) = \cos(\frac{\theta}{2}), b(\theta) = 0, c(\theta) = 0, d(\theta) = -i\sin(\frac{\theta}{2})$.

   (b) $\rho_1 = \mathrm{tr}\,_2(|\psi(\theta)\rangle\langle\psi(\theta)| = \langle 0|_2|\psi(\theta)\rangle\langle\psi(\theta)||0\rangle_2 + \langle 1|_2|\psi(\theta)\rangle\langle\psi(\theta)||1\rangle_2 = \cos^2(\frac{\theta}{2})|0\rangle\langle 0| + \sin^2(\frac{\theta}{2})|1\rangle\langle 1|$

   (c) This reduced state of the first qubit is pure only when $\cos(\frac{\theta}{2}) = 0$ or $\sin(\frac{\theta}{2}) = 0$, *i.e.*, when $\theta = \pi$ or $\theta = 0$. These values of θ correspond to values of $U(\theta)$ that are separable, meaning that it implements a separate unitary gate on the two qubits. When $\theta = \pi$ this is a σ_x rotation on each qubit, and when $\theta = 0$ it is the identity on both qubits.

3. ***Generalized measurement.*** In the lecture notes we went through how a generalized measurement can be implemented by coupling to an ancilla and then doing projective measurements on the ancilla degrees of freedom. In this problem you will work out what the POVM elements are for a particular implementation of a type of generalized measurement called a *weak measurement.*

   Let the main system be a qubit in an arbitrary state $|\psi\rangle$. This qubit is coupled to another qubit that constitutes the ancilla, according to the circuit



   The projective measurement of the ancilla qubit is in the computational basis, and the H represents a Hadamard gate. The coupling unitary has the form:

$$U(\theta) = e^{-i\frac{\theta}{2}Z\otimes Y}, \qquad 0 \leq \theta \leq 2\pi$$

   (a) $U(\theta) = e^{-i\frac{\theta}{2}Z\otimes Y} = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})Z \otimes Y$. Therefore,

$$U(\theta)|\psi\rangle \otimes |0\rangle = \cos(\frac{\theta}{2})|\psi\rangle \otimes |0\rangle + \sin(\frac{\theta}{2})Z|\psi\rangle \otimes |1\rangle$$

   Then applying the Hadamard to the ancilla qubit results in the pre measurement state

$$\begin{aligned} |\Psi_{\mathrm{pre-meas}}\rangle \;&=\; \cos(\frac{\theta}{2})|\psi\rangle \otimes \Big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\Big) + \sin(\frac{\theta}{2})Z|\psi\rangle \otimes \Big(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\Big) \\ & \frac{1}{\sqrt{2}}\Big[\Big(\cos(\frac{\theta}{2})|\psi\rangle + \sin(\frac{\theta}{2})Z|\psi\rangle\Big) \otimes |0\rangle + \Big(\cos(\frac{\theta}{2})|\psi\rangle - \sin(\frac{\theta}{2})Z|\psi\rangle\Big) \otimes |1\rangle\Big] \end{aligned}$$

(b) The probabilities for the two outcomes when measuring the ancilla qubit are:

$$\Pr\{0\} = \mathrm{tr}\left(I \otimes |0\rangle\langle 0|\ |\Psi_{\mathrm{pre-meas}}\rangle\langle\Psi_{\mathrm{pre-meas}}|\right) = \frac{1}{2}\left(1+\sin(\theta)\langle\psi|\sigma_z|\psi\rangle\right)$$
$$\Pr\{1\} = \mathrm{tr}\left(I \otimes |1\rangle\langle 1|\ |\Psi_{\mathrm{pre-meas}}\rangle\langle\Psi_{\mathrm{pre-meas}}|\right) = \frac{1}{2}\left(1-\sin(\theta)\langle\psi|\sigma_z|\psi\rangle\right)$$

(c) By directly using the definition of the measurement operators and the form of the post measurement state derived in part (a), we get:

$$\begin{aligned} M_0 &= \frac{1}{\sqrt{2}}\Big[\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})Z\Big] \\ M_1 &= \frac{1}{\sqrt{2}}\Big[\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})Z\Big] \end{aligned} \tag{1}$$

(d) By direct computation,

$$E_0 = \frac{I}{2} + \sin(\theta)Z$$
$$E_0 = \frac{I}{2} - \sin(\theta)Z$$

Adding these together clearly results in $\sum_i E_i = I$.

(e) Returning to the expressions for the measurement operators on Eq. (1), we see that they are already diagonal in the computational basis. Therefore we can simply read off the values of a_i and b_i. Explicitly writing the measurement operators in the form required,

$$\begin{aligned} M_0 &= \frac{1}{\sqrt{2}}\Big(\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})\Big)|0\rangle\langle 0| + \frac{1}{\sqrt{2}}\Big(\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})\Big)|1\rangle\langle 1| \\ M_1 &= \frac{1}{\sqrt{2}}\Big(\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})\Big)|0\rangle\langle 0| + \frac{1}{\sqrt{2}}\Big(\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})\Big)|1\rangle\langle 1| \end{aligned} \tag{2}$$

4. ***CHSH inequality.*** This is a direct computation of the correlation functions under the given quantum state. For example, $E(P,R) = \mathrm{tr}\left(\rho P \otimes R\right)$. Computing these four correlations functions in terms of p, we get:

$$\begin{aligned} E(P,R) &= p-1 \\ E(Q,R) &= \frac{p-1}{\sqrt{2}} \\ E(P,S) &= \frac{p-1}{\sqrt{2}} \\ E(Q,S) &= 0 \end{aligned} \tag{3}$$

Therefore the CHSH quantity is

$$|CHSH| = |(p-1) + \frac{p-1}{\sqrt{2}} + \frac{p-1}{\sqrt{2}}| = \frac{\sqrt{2}+2}{\sqrt{2}}(1-p) \tag{4}$$

This quantity is > 2 (*i.e.*, the CHSH inequality is violated) when $p \leq 1 - \frac{2\sqrt{2}}{\sqrt{2}+2} \approx 0.1716$. Therefore we see that one doesn't need to add a lot of noise to the maximally entangled state $\left|\psi^{AB}\right\rangle$ before it fails to violate the CHSH inequality.